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NEW SYMMETRIES AND LIE ALGEBRAS OF THE GENERAL KdV EQUATION

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The strong symmetry of the general KdV equation is factorized to a simple form and then the inverse strong symmetry is obtained explicitly. Acting a strong symmetry of the general KdV equation on the trivial symmetry and the known τ_c symmetry, we obtain four new sets of symmetries of the general KdV equation. All these sets of symmetries constitute an infinite dimensional Lie algebra.

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Symmetry theory plays an important role in every field of physics. For an integrable model, using any one symmetry, one can get an exact group invariant solution using the standard classical Lie approach.[1] It has been shown that almost all the well-known integrable nonlinear evolution equation possess infinite symmetries, and these symmetries usually constitute some infinite dimensional Lie algebras.[2-4] A powerful tool to get new symmetries is to use the strong symmetry (or recursion operator) introduced first by Olver.[5] When a strong symmetry is obtained, one can get a set of symmetries by acting the strong symmetry on some seed symmetry. Recently, Lou[6-10] developed a simple method to factorize the strong symmetry such that the inverse strong symmetry can be obtained explicitly at the same time, and found that there exist much more abundant symmetries for various integrable models such as KdV, MKdV, Candrey-Dodd-Gibbon-Sowada-Kortera, Kaup-Kupershmidt and Kawamoto-type equations. In this article, we would like to give more information about the symmetries and the related infinite dimensional Lie algebras for the general KdV equation

$$u_t = u_{xxx} + 6uu_x + 6f(t)u + x(f' - 12f^2)\,, \tag{1}$$

where $f(t)$ is an arbitrary function of t, the superscript "$'$"$\equiv \mathrm{d}/\mathrm{d}t$.

Tian[11] had obtained one strong symmetry and two sets of symmetries of Eq.(1). The strong symmetry is of the form

$$\Phi = g^{-1}[\mathrm{D}^2 + 4(u - xf) + 2(u_x - f)\mathrm{D}^{-1}]\,, \tag{2}$$

where $g = \exp(\int 12f\mathrm{d}t), \mathrm{D} = \mathrm{d}/\mathrm{d}x, \mathrm{D}^{-1}\mathrm{D} = \mathrm{D}\mathrm{D}^{-1} = I$. The two sets of symmetries are

$$\sigma_n^{(\mathrm{o})} = \Phi^n \sigma_0^{(\mathrm{o})}\,, \quad n = 0, 1, 2, \cdots\,, \tag{3}$$

$$\tau_n^{(o)} = \Phi^n \tau_o^{(o)} , \quad n = 0, 1, 2, \cdots , \tag{4}$$

where

$$\sigma_o^{(o)} = g^{-1/2}(u_x - f) , \tag{5}$$

$$\tau_o^{(o)} = 3(g^{-1/2} \int g^{3/2} \mathrm{d}t)(u_x - f) + (1/2)g . \tag{6}$$

After factorizing the strong symmetry Φ, we obtain

$$\begin{aligned} \Phi &= g^{-1}(\mathrm{D} - h)\mathrm{D}(\mathrm{D} + h)\mathrm{D}^{-1} \\ &= g^{-1}\exp(\mathrm{D}^{-1}h)\mathrm{D}\ \exp(-\mathrm{D}^{-1}h)\mathrm{D}\ \exp(-\mathrm{D}^{-1}h)\mathrm{D}\ \exp(\mathrm{D}^{-1}h)\mathrm{D}^{-1} , \end{aligned} \tag{7}$$

where h and $(u - xf)$ satisfy the following relation:

$$2h_x - h^2 = 4(u - xf) . \tag{8}$$

From Eq.(7), we get the inverse strong symmetry of Eq.(1)

$$\Phi^{-1} = g\mathrm{D}\ \exp(-\mathrm{D}^{-1}h)\mathrm{D}^{-1}\ \exp(\mathrm{D}^{-1}h)\mathrm{D}^{-1}\ \exp(\mathrm{D}^{-1}h)\mathrm{D}^{-1}\ \exp(-\mathrm{D}^{-1}h) . \tag{9}$$

Let

$$\exp(-\mathrm{D}^{-1}h) = \Psi^2 , \tag{10}$$

then Φ and Φ^{-1} become

$$\Phi = g^{-1}\Psi^{-2}\mathrm{D}\Psi^2\mathrm{D}\Psi^2\mathrm{D}\Psi^{-2}\mathrm{D}^{-1} , \tag{11}$$

$$\Phi^{-1} = g\mathrm{D}\Psi^2\mathrm{D}^{-1}\Psi^{-2}\mathrm{D}^{-1}\Psi^{-2}\mathrm{D}^{-1}\Psi^2 , \tag{12}$$

where Ψ is related to $(u - xf)$ by the Schrödinger equation with zero eigenvalue

$$\Psi_{xx} + (u - xf)\Psi = 0 . \tag{13}$$

The symmetry definition equation for Eq.(1) reads

$$\mathrm{d}\sigma/\mathrm{d}t = K'[\sigma] = (\mathrm{D}^3 + 6u\mathrm{D} + 6u_x + 6f)\sigma . \tag{14}$$

From Eq.(14), we can show that $\sigma \equiv 0$ is a trivial symmetry. In order to find some new seed symmetries, by acting Φ^{-1} on the trivial symmetry $\sigma \equiv 0$, we have

$$\sigma_{-1} \equiv \Phi^{-1}0 = C_1(t)\mathrm{D}\Psi^2 + C_2(t)\mathrm{D}\Psi^2\mathrm{D}^{-1}\Psi^{-2} + C_3(t)\mathrm{D}\Psi^2 D^{-1}\Psi^{-2}\mathrm{D}^{-1}\Psi^{-2} , \tag{15}$$

if we take

$$\mathrm{D}^{-1}0 = C(t) , \tag{16}$$

where $C(t)$ is an arbitrary function of t, which is determined by Eq.(14).

Fixing the integration functions of $C_1(t), C_2(t), C_3(t)$, and the function Ψ, the result tells us that

$$\sigma_o^{(1)} = g^{1/2}\mathrm{D}\Psi^2 , \tag{17}$$

$$\sigma_o^{(2)} = g^{1/2}\mathrm{D}\Psi^2\mathrm{D}^{-1}\Psi^{-2} , \tag{18}$$

$$\sigma_o^{(3)} = g^{1/2}\mathrm{D}\Psi^2\mathrm{D}^{-1}\Psi^{-2}\mathrm{D}^{-1}\Psi^{-2} \tag{19}$$

are all symmetries of Eq.(1) when Ψ also satisfies

$$\Psi_t = 2(2xf + u)\Psi_x - (u_x - f)\Psi , \tag{20}$$

in addition to satisfying Eq.(13).

It is interesting that the integrability condition of Eqs.(13) and (20), $\Psi_{xxt} = \Psi_{txx}$, is just the general KdV equation (1).

By acting Φ^{-1} on the $\tau_o^{(o)}$ and fixing the integrable function, we obtain that

$$\tau_o^{(1)} = \Phi^{-1}\tau_o^{(o)} = g^2\mathrm{D}\Psi^2\mathrm{D}^{-1}\Psi^{-2}\mathrm{D}^{-1}\Psi^{-2}\mathrm{D}^{-1}\Psi^2 \tag{21}$$

is also symmetry of Eq.(1) with Ψ being given by Eqs.(13) and (20).

In fact, symmetries (17), (18), (19) and (20) of Eq.(1) can be obtained by transformation theorem in Ref.[12] and the known symmetries of KdV equation in Ref.[6].

Now acting Φ and Φ^{-1} on the seed symmetries (17)-(20), we get four new sets of infinite many symmetries

$$\sigma_{-n}^{(1)} = \Phi^{-n}\sigma_o^{(1)} , \qquad n = 0, 1, 2, \cdots , \tag{22}$$

$$\sigma_{-n}^{(2)} = \Phi^{-n}\sigma_o^{(2)} , \qquad n = 0, 1, 2, \cdots , \tag{23}$$

$$\sigma_{-n}^{(3)} = \Phi^{-n}\sigma_o^{(3)} , \qquad n = 0, 1, 2, \cdots , \tag{24}$$

$$\tau_{-n}^{(1)} = \Phi^{-n}\tau_o^{(1)} , \qquad n = 0, 1, 2, \cdots . \tag{25}$$

It is necessary to point out that all these sets of symmetries are only half hierarchies because one can directly verify that

$$\Phi^{-1}\sigma_o^{(o)} = 0 , \tag{26}$$

$$\Phi\sigma_o^{(1)} = \Phi\sigma_o^{(2)} = \Phi\sigma_o^{(3)} = 0 . \tag{27}$$

Tian[11] had obtained infinite Lie algebras satisfied by symmetries $\sigma_n^{(o)}$ and $\tau_n^{(o)}$ as follows:

$$[\sigma_m^{(o)}, \sigma_n^{(o)}] = 0 , \quad n, m = 0, 1, 2, \cdots , \tag{28}$$

$$[\sigma_m^{(o)}, \tau_n^{(o)}] = (2m + 1)\sigma_{-m-n-1}^{(o)} , \quad n, m = 0, 1, 2, \cdots , \tag{29}$$

$$[\tau_m^{(o)}, \tau_n^{(o)}] = 2(m-n)\tau_{m+n-1}^{(o)} , \quad n, m = 0, 1, 2, \cdots . \tag{30}$$

We can also prove that new symmetries of Eq.(1) constitute similarly infinite Lie algebras

$$[\sigma_m^{(o)}, \sigma_{-n}^{(1)}] = 0 , \quad n, m = 0, 1, 2, \cdots , \tag{31}$$

$$[\sigma_m^{(o)}, \sigma_{-n}^{(2)}] = 0 , \quad n, m = 0, 1, 2, \cdots , \tag{32}$$

$$[\sigma_m^{(o)}, \sigma_{-n}^{(3)}] = 0 , \quad n, m = 0, 1, 2, \cdots , \tag{33}$$

$$[\sigma_{-m}^{(1)}, \sigma_{-n}^{(1)}] = 0 , \quad n, m = 0, 1, 2, \cdots , \tag{34}$$

$$[\sigma_{-m}^{(2)}, \sigma_{-n}^{(2)}] = 0 , \quad n, m = 0, 1, 2, \cdots , \tag{35}$$

$$[\sigma_{-m}^{(3)}, \sigma_{-n}^{(3)}] = 0 , \quad n, m = 0, 1, 2, \cdots , \tag{36}$$

$$[\sigma_{-m}^{(1)}, \sigma_{-n}^{(2)}] = -2\sigma_{-m-n}^{(3)} , \quad n, m = 0, 1, 2, \cdots , \tag{37}$$

$$[\sigma_{-m}^{(1)}, \sigma_{-n}^{(3)}] = -2\sigma_{-m-n-1}^{(1)} , \quad n, m = 0, 1, 2, \cdots , \tag{38}$$

$$[\sigma_{-m}^{(2)}, \sigma_{-n}^{(3)}] = 2\sigma_{-m-n-1}^{(2)} , \quad n, m = 0, 1, 2, \cdots , \tag{39}$$

$$[\sigma_{-m}^{(o)}, \tau_{-n}^{(1)}] = (2m+1)\sigma_{m-n-1}^{(o)} , \quad n, m = 0, 1, 2, \cdots , \tag{40}$$

$$[\sigma_{-m}^{(1)}, \tau_{-n}^{(1)}] = -2(m+1)\sigma_{-m-n-1}^{(1)} , \quad n, m = 0, 1, 2, \cdots , \tag{41}$$

$$[\sigma_{-m}^{(2)}, \tau_{-n}^{(1)}] = -2m\sigma_{-m-n-1}^{(2)} , \quad n, m = 0, 1, 2, \cdots , \tag{42}$$

$$[\sigma_{-m}^{(3)}, \tau_{-n}^{(1)}] = -2(m+1)\sigma_{n-m-1}^{(3)} , \quad n, m = 0, 1, 2, \cdots , \tag{43}$$

$$[\sigma_{-m}^{(1)}, \tau_{n}^{(0)}] = -2(m+1)\sigma_{n-m-1}^{(1)} , \quad n, m = 0, 1, 2, \cdots , \tag{44}$$

$$[\sigma_{-m}^{(2)}, \tau_{n}^{(0)}] = -(2m+1)\sigma_{n-m-1}^{(2)} , \quad n, m = 0, 1, 2, \cdots , \tag{45}$$

$$[\sigma_{-m}^{(3)}, \tau_{n}^{(0)}] = -2(m+n)\sigma_{n-m-1}^{(3)} , \quad n, m = 0, 1, 2, \cdots , \tag{46}$$

$$[\tau_m^{(0)}, \tau_{-n}^{(1)}] = 2(m+n)\tau_{-n-m-1}^{(1)} , \quad n, m = 0, 1, 2, \cdots , \tag{47}$$

$$[\tau_{-m}^{(1)}, \tau_{-n}^{(1)}] = 2(n-m)\tau_{-m-n-1}^{(1)} , \quad n, m = 0, 1, 2, \cdots , \tag{48}$$

It is also well known that every symmetry of the integrable model defines another integrable model

$$u_t = \sigma , \tag{49}$$

which possesses some similar integrable properties. Starting from the symmetries of Eq.(1), we get four new hierarchies of integrable models

$$u_t = \Phi^{-n}\sigma_o^{(1)} , \qquad n = 0, 1, 2, \cdots , \tag{50}$$

$$u_t = \Phi^{-n}\sigma_o^{(2)} , \qquad n = 0, 1, 2, \cdots , \tag{51}$$

$$u_t = \Phi^{-n}\sigma_0^{(3)} , \qquad n = 0, 1, 2, \cdots , \tag{52}$$

$$u_t = \Phi^{-n}\tau_0^{(1)} , \qquad n = 0, 1, 2, \cdots , \tag{53}$$

where u and Ψ satisfy Eqs.(13) and (20).

In summary, there exist four sets of time-independent symmetries and two sets of time-dependent symmetries of the general KdV equation. Four of them are new and can be expressed as some integrations of the Jost functions of the general KdV equation. And then one can get six hierarchies of the integrable equation. Four new hierarchies are integro-differential equations and two old ones are differential equations. In addtion, these symmetries constitute an infinite dimensional Lie algebra.

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