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THE UNIVERSAL TRANSITION OF PLATEAU STRUCTURE OF LYAPUNOV EXPONENTS*

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The universal transition of Lyapunov exponents between conservative limit and dissipative limit of nonlinear dynamical system is studied. It is discovered numerically and proved analytically that for homogeneous dissipative two-dimensional maps, along the equal dissipation line in parameter space, the Lyapunov exponents of attractor orbits possess a plateau structure and strict symmetry about its plateau value. The ratios between the plateau width and the stable window width of period 1–4 orbits for Henon map are calculated. The result shows that the plateau structure of Lyapunov exponents remains invariant for the attractor orbits belonging to a period doubling bifurcation sequence. This fact reveals a new universal transition behavior between order and chaos when the dissipation of the dynamical system is weakened to zero.

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I. INTRODUCTION

The long term asymptotic dynamical behavior of dissipative system is quite different from conservative system. However, the conservative system can be regarded as the limit of dissipative system when the dissipation is weakend finally to zero. Hence it is an interesting basic problem of nonlinear dynamics to understand how the regular and strange attractors in dissipative system become three types of orbit in conservative system, namely, periodic orbit, KAM invariant torus and stochastic orbits, as the dissipation approaches zero, and furthermore, to study the universality of transition between dissipative chaos and conservative chaos.[1,2]

Using the method of mapping on Poincaré surface section in three-dimensional phase space, we can derive area-preserving map from measure-preserving flow, and two-dimensional dissipative map from dissipative flow. Hence, any two-dimensional dissipative map with at least two variable parameters representing dissipation and other nonlinearity such as nonintegrable perturbation strength may become the dynamical model for study of universal

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transition between dissipative system and conservative system. Henon map is a typical example of two-dimensional dissipative map[3]

$$x_{n+1} = 1 - Ax_n^2 + y_n \ ,$$
$$y_{n+1} = -Jx_n \ , \tag{1}$$

where the dissipation parameter J equals exactly the value of Jacobian matrix determinant of the map

$$J = \det \frac{\partial(x_{n+1}, y_{n+1})}{\partial(x_n, y_n)} \ . \tag{2}$$

Hence, for given J value, the map is homogeneous dissipative, in other words, the contraction rate of phase area is constant at every point of phase plane.

We have pointed out in Refs.[1] and [2] that there exists a continuous corridor of stability window extended from $J=1$ (the conservative limit) to $J=0$ (the strongest dissipative limit) in parameter A-J space for any periodic orbit of homogeneous dissipative two-dimensional map. Although the width of stability corridor becomes more and more narrow rapidly when $J \to 0$ and the period order increases, the corridor will not be interrupted. All the stability corridors of periodic orbits, namely, the parameter region for all regular attractors, still cannot cover the whole parameter space. Their complement is the parameter space occupied by strange attractors or strange repellers. The band structure of strange attractor will be truncated universally as J increases from 0 to 1, that is, the strange attractor with 2^n pieces of chaotic bands will merge into one piece strange attractor via crisis at a critical Jacobian value J_n, in other words, for $J > J_n$, all the strange attractors with $2^i (i = 1, 2, \cdots, n)$ chaotic pieces are extinguished, and the value J_n satisfies universal scaling law

$$\lim_{n\to\infty} \frac{J_n - J_{n-1}}{J_{n+1} - J_n} = 2 \ . \tag{3}$$

In the present paper, we study the crossover behavior of Lyapunov exponents in transition from dissipation limit to conservation limit for homogeneous dissipative two-dimensional mapping. We discover that along the constant J line, the Lyapunov exponents of all the regular attractors show a plateau structure, the plateau value only depends on J and there is a strict symmetry relation about this plateau for two Lyapunove exponents λ_1 and $\lambda_2 (\lambda_1 > \lambda_2)$ of any attractor no matter whether it is periodic, quasi periodic or strange. Furthermore, we study variations of the ratio of plateau width and stable window width for orbits that belong to period-doubling sequence as J value changes. A new type of invariant function relation has been found which reveals novel universal transition behavior of nonintegrable dissipative system when it approaches conservative limit.

II. THE PLATEAU STRUCTURE AND SYMMETRY OF LYAPUNOV EXPONENT

For ordinary one-dimensional differentiable noninvertible mapping

$$x_{n+1} = f(\mu, x_n) \tag{4}$$

(for example, $f(\mu, x) = \mu - x^2$ for Logistic map) we know that the Lyapunov exponent of any period-N orbit will have an abyss with infinite depth at its superstable point μ_N for which

$$f^N(\mu_N, x^*) = 0 \ , \tag{5}$$

where x^* is the point at which the function f^N attains its extremum, that is, λ_1 reaches its minimum value which is minus infinite at μ_N

$$\lambda_1(\mu_N) = \lambda_{1,\min} = -\infty \ .$$

However, for homogeneous dissipative two-dimensional mapping, if Jacobian value J is given ($J > 0$), the largest Lyapunov exponent λ_1 of any periodic orbit will not present an abyss with infinite depth, its minimum is a finite value and may present a plateau structure for other controlled nonlinearity parameter. This phenomenon was pointed out by Kaplan for the first time.[4]

The plateau value of Lyapunov exponents is

$$\lambda_p = \frac{1}{2} \ln J \ , \tag{6}$$

which only depends on J value. The plateau is both the minimum of the largest Lyapunov exponent λ_1 and the maximum of the smaller exponent λ_2, and becomes a strict symmetry axis for λ_1 and λ_2 at which λ_1 comes into contact with λ_2, that is

$$\lambda_1 \geq \lambda_p \ , \quad \lambda_2 \leq \lambda_p \ ,$$

$$\lambda_1 - \lambda_p = \lambda_p - \lambda_2 \ . \tag{7}$$

In Fig. 1 we show two Lyapunov exponents λ_1 and λ_2 as the function of nonintegrability strength parameter A for Henon mapping (1) with $J = 0.3$. An apparent plateau structure with the plateau value

$$\lambda_p = (1/2) \ln 0.3 = -0.6019864$$

and a strict symmetric relation of λ_1 and λ_2 around λ_p can be seen.

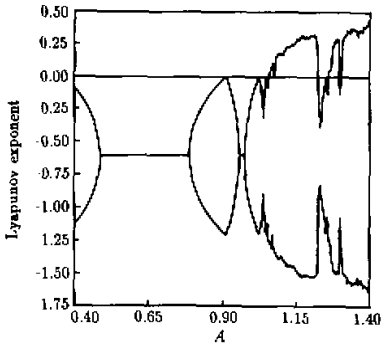

Fig. 1. Lyapunov exponents λ_1 and λ_2 for Henon map ($J = 0.3$).

Now we give an analytical proof for the plateau and symmetry structure of Lyapunov exponents for general homogeneous dissipative two-dimensional mapping

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = T \begin{bmatrix} x_n \\ y_n \end{bmatrix} , \tag{8}$$

of which the linear Jacobian matrix is

$$M_n = \frac{\partial(x_{n+1}, y_{n+1})}{\partial(x_n, y_n)} . \tag{9}$$

The homogeneous dissipation means that

$$J = \det M_n = \text{const}, \quad |J| \leq 1 .$$

For simplicity, we assume that J is positive. The Jacobian matrix of period-N orbit is

$$M^{(N)} \equiv M_N \cdot M_{N-1} \cdots M_2 \cdot M_1 . \tag{10}$$

Its determinant is

$$\det M^{(N)} = \prod_{i=1}^{N} \det M_i = J^N . \tag{11}$$

Two eigenvalues σ_1 and σ_2 of matrix $M^{(N)}$ must satisfy the following equation:

$$\sigma^2 - (\mathrm{Tr} M^{(N)})\sigma + \det M^{(N)} = 0 . \tag{12}$$

If we define F_N as half of the trace of matrix $M^{(N)}$

$$F_N = \frac{1}{2} \mathrm{Tr} M^{(N)} , \tag{13}$$

the solution of eigenvalue can be written as

$$\sigma_{1,2}^{(N)} = F_N \pm \sqrt{F_N^2 - J^N} . \tag{14}$$

Suppose

$$|\sigma_1^{(N)}| \geq |\sigma_2^{(N)}| , \tag{15}$$

the Lyapunov exponent spectrum of period-N orbit can be obtained from the eigenvalues as

$$\lambda_1 = \frac{1}{N} \ln |\sigma_1^{(N)}|, \quad \lambda_2 = \frac{1}{N} \ln |\sigma_2^{(N)}| . \tag{16}$$

The sum of the two exponents is

$$\lambda_1 + \lambda_2 = \frac{1}{N} \ln |\sigma_1^{(N)} \cdot \sigma_2^{(N)}| = \ln J , \tag{17}$$

from which the following symmetry relation results:

$$\lambda_1 - \frac{1}{2} \ln J = \frac{1}{2} \ln J - \lambda_2 . \tag{18}$$

Hence, Eqs. (6) and (7) are proved to be true.

For chaotic orbits, Lyapunov exponents should be defined as

$$\lambda_{1,2} = \lim_{N \to \infty} \frac{1}{N} \ln |\sigma_{1,2}^{(N)}| , \tag{19}$$

where $\sigma_{1,2}^{(N)}$ are the two eigenvalues of the linearized matrix of the N-th iterated map $M^{(N)}$, hence the above conclusion is still correct for $N \to \infty$. Therefore, the two Lyapunov exponents of any attractor along the equal J line, no matter whether it is regular or chaotic, are strictly symmetric about the value of λ_p for any two-dimensional homogeneou dissipative map.

Now, we can prove that λ_p is both the minimum of λ_1 and the maximum of λ_2, and present a plateau structure for other nonintegrability parameters.

In face, eigenequation (12) and the solution form of eigenvalue (14) indicate that when non-integrability parameter varies, two eigenvalues $\sigma_{1,2}^{(N)}$ in the stable windows of period-N orbit will move on the complex σ plane usually in the way as shown in Fig. 2.

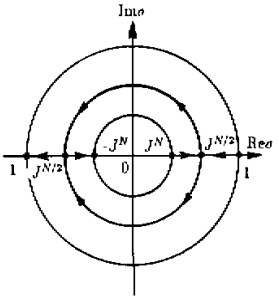

Fig. 2. An illustration of movement of eigenvalue on the complex plane for stable period-N orbit.

In Fig. 2, we can find that the onset boundary point of stable window of period-N orbit is $\sigma_1 = 1$, $\sigma_2 = J^N$, from which the two eigenvalues move along the positive real axis, and merge at

$$\sigma_1 = \sigma_2 = J^{N/2} ,$$

then rotate as conjugate complex along a circle with the center at the origin and the radius equal to $J^{N/2}$ until meet at $-J^{N/2}$. Afterwards, they part again along negative real axis and approach the upper boundary of stable window at $\sigma_1 = -1$, $\sigma_2 = -J^N$ where the period-N orbit destabilizes via period doubling bifurcation.

Two conjugate complex eigenvalues $\sigma_{1,2}$ yield equal Lyapunov exponents when they rotate along the circle with radius $J^{N/2}$ as the parameter varies, since the moduli of complex $\sigma_{1,2} = J_N$ are equal and keep constant.

For homogeneous dissipative two-dimensional maps with given constant J value, the small circle along which the eigenvalues rotate corresponds to a value range of other nonlinear controlled parameters, hence there is a plateau structure of Lyapunov exponents for the variable parameter present. The value of Lyapunov exponent on plateau is given by Eq. (6) according to plateau's definition Eq. (16).

Since σ_1 and σ_2 are located respectively outside and inside the circle with radius $J^{N/2}$, the plateau exponent value must be the minimum of λ_1 and the maximum of λ_2

$$\lambda_{1,\min} = \lambda_{2,\max} = \lambda_p . \tag{20}$$

The rotation behavior of the two eigenvalues to which the plateaus of Lyapunov exponents correspond can be represented explicitly as[4]

$$\sigma_{1,2}^{(N)} = F_N \pm \mathrm{i}\sqrt{J^N - F_N^2} = J^{N/2} e^{\pm \mathrm{i}\theta} , \tag{21}$$

where the argument θ is

$$\theta = \arctan \frac{\sqrt{J^N - F_N^2}}{F_N} . \tag{22}$$

Put $|\sigma| \leq 1$ in the eigenequation (12), the stable window of period-N orbit can be obtained

$$|F_N| \leq \frac{1}{2}(1 + J^N) . \tag{23}$$

While the rotation region of eigenvalue, namely, the plateaus of Lyapunov exponents can be derived from

$$|F_N| \leq J^{N/2} . \tag{24}$$

For area-preserving maps, we have $\lambda_1 + \lambda_2 = 0$ since $J = 1$. This means that two Lyapunov exponents are symmetric about $\lambda = 0$ axis, their absolute values are equal. Hence, for all stable (elliptic) periodic orbits in area-preserving maps, we have $\lambda_1 = \lambda_2 = 0$, that is, the whole stable window is the plateau of Lyapunov exponent, while for stochastic orbit we have $\lambda_1 > 0$, $\lambda_2 < 0$, and $\lambda_2 = -\lambda_1$.

For $J \to 0$, the two-dimensional dissipative map reduces to a one-dimensional noninvertible map and the plateau value $\lambda_p \to -\infty$. This means that at the strongest dissipation limit, λ_1 of period-N orbit will present an abyss with infinite depth, the plateau width shrinks to zero, and λ_2 cannot exist.

For the chaotic orbits of strange attractor of general homogeneous dissipative mapping, although the plateau structure will not present again, the symmetry of λ_1 and λ_2 about plateau value λ_p still remains. Hence the Lyapunov fractal dimension of strange attractor along the equal dissipation line[5]

$$D_L \equiv 1 - \frac{\lambda_1}{\lambda_2} = 1 + \frac{\lambda_1}{|\lambda_2|}$$

may be obtained from only the largest Lyapunov exponent λ_1

$$D_L = 1 + \frac{\lambda_1}{\lambda_1 + |\ln J|} . \tag{25}$$

III. THE WIDTH OF LYAPUNOV EXPONENT PLATEAU OF PERIODIC ORBIT FOR HENON MAP

The plateau values λ_p are the same for all the orbits along J line. However, the parameter width of plateaus are different for different period N. The boundary of every plateau can be determined by Eq. (24). The plateau must be a part of stable window. The variation scenario of the ratio of plateau width to stable window width as the dissipation parameter J changes, contains the dynamical information of the system about transition between conservation and dissipation.

The stable windows and the Lyapunov exponent plateaus of periodic orbits for Henon map can be calculated as follows.

Via coordinate transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -Ay/J \\ Ax \end{bmatrix} \tag{26}$$

the mapping Eq. (1) can be transformed to the map

$$x_{N+1} = y_N ,$$

$$y_{N+1} = A - Jx_N - y_N^2 \ . \tag{27}$$

Its fixed point attractor is (x_1, y_1), where

$$x_1 = y_1 = \frac{1}{2}[-(1+J) \pm \sqrt{(1+J)^2 + 4A}\ ] \ . \tag{28}$$

From Jacobian matrix, we obtain $F_1 = -y_1$. From $|F_1| \leq \dfrac{1+J}{2}$, we obtain stable window of period-1 orbit

$$-\frac{1}{4}(1+J)^2 \leq A \leq \frac{3}{4}(1+J)^2 \tag{29}$$

from which we obtain the width of stable window

$$W_1 = (1+J)^2 \ . \tag{30}$$

And $|F_1| \leq J^{1/2}$ yields the plateau region of Lyapunov exponent as

$$J - (1+J)J^{1/2} \leq A \leq J + (1+J)J^{1/2} \ , \tag{31}$$

which gives the width of plateau as

$$P_1 = 2(1+J)J^{1/2} \ . \tag{32}$$

Then we can give the ratio of the A-width of the plateau divided by the A-width of the stability window for the period-1 orbit of the Henon map as[4]

$$\alpha_1(J) \equiv \frac{P_1(J)}{W_1(J)} = \frac{2J^{1/2}}{1+J} \ . \tag{33}$$

The above result is applicable for $0 \leq J \leq 1$. For negative value of J, it can be proved that the value of α_1 is equal to $\alpha_1(|J|)$. Hence for periodic orbits of Henon map, although the variations of stable window with J is not symmetric about $J = 0$, the width ratio $\alpha_N(J)$ is strictly symmetrical.

For period-2 attractor, we have

$$F_2 = -J + 2y_1y_2 = 2(1+J)^2 - J - 2A \ , \tag{34}$$

from which we obtain the stable window of the orbit

$$\frac{3}{4}(1+J)^2 \leq A \leq \frac{1}{4}(5J^2 + 6J + 5) \tag{35}$$

and the plateau region of Lyapunov exponent

$$(1+J)^2 - J \leq A \leq (1+J)^2 \ . \tag{36}$$

Hence the ratio of widths is[4]

$$\alpha_2(J) = \frac{2J}{1+J^2} . \tag{37}$$

For period-3 attractor, using the analytical method in Ref. [6], we can obtain stable window. The discriminant function determining stable window and plateau width is

$$F_3 = J(y_1 + y_2 + y_3) - 4y_1y_2y_3 . \tag{38}$$

Substitute the solution of period-3 orbit into function F_3, we obtain

$$\begin{aligned} F_3 &= \frac{1+J}{2}(8 - 11J + 8J^2) - 2A(1+J) \\ &\pm (3J - 4A)\sqrt{A - \frac{1}{4}(7 - 10J + 7J^2)} . \end{aligned} \tag{39}$$

Then we can determine the stability window of the orbit from

$$|F_3| = \frac{1}{2}(1+J^3)$$

as

$$\frac{1}{4}(7 - 10J + 7J^2) \le A \le A_s(J) , \tag{40}$$

where $A_s(J)$ is the real root of the following cubic equation:

$$\begin{aligned} & 64A_s^3 - 32(4 - J + 4J^2)A_s^2 \\ + \; & 36(2 + 6J - 7J^2 + 6J^3 + 2J^4)A_s \\ - \; & 9(9 - 6J + 2J^2 + 10J^3 + 2J^4 - 6J^5 + 9J^6) = 0 , \end{aligned} \tag{41}$$

A_s can be represented explicitly as

$$A_s = \frac{4 - J + 4J^2}{6} + \sqrt[3]{-\frac{q_s}{2} + \sqrt{\Delta_s}} + \sqrt[3]{-\frac{q_s}{2} - \sqrt{\Delta_s}} , \tag{42}$$

where

$$\begin{aligned} \Delta_s &= (\frac{q_s}{2})^2 + (\frac{p_s}{3})^3 > 0 , \\ p_s &= -\frac{10 - 194J + 321J^2 - 194J^3 + 10J^4}{48} \end{aligned} \tag{43}$$

$$\begin{aligned} q_s \;=\; & \frac{1}{1782}(1915 - 5790J + 7962J^2 - 8032J^3 \\ + \; & 7962J^4 - 5790J^5 + 1915J^6) . \end{aligned} \tag{44}$$

The plateau region of Lyapunov exponent of period-3 orbit can be obtained similarly from $|F_3| \le J^{3/2}$ as

$$A_{p1} \le A \le_{p2} . \tag{45}$$

The boundary value A_{p1} and A_{p2} of plateau region are the real roots of the following cubic equation respectively:

$$\begin{aligned} & 16A^3 - 8(4 - J + 4J^2)A^2 \\ + \; & [16 + 52J - 63J^2 + 52J^3 + 16J^4 \mp 4(1+J)J^{\frac{3}{2}}]A \\ - \; & [16 - 12J + 6J^2 + 15J^3 + 6J^4 - 12J^5 + 16J^6 \\ \mp \; & (1+J)(8 - 11J + 8J^2)J^{\frac{3}{2}}] = 0 , \end{aligned} \tag{46}$$

where the upper and lower signs ($\mp$) correspond to the low and high limits: A_{p1} and A_{p2} respectively. Using Cardan formula, A_{p1} and A_{p2} can be represented explicitly as

$$A_p = \frac{4 - J + 4J^2}{6} + (\sqrt{\Delta_p} - \frac{q_p}{2})^{1/3} - (\sqrt{\Delta_p} + \frac{q_p}{2})^{1/3} , \tag{47}$$

where

$$\Delta_p = (\frac{q_p}{2})^2 + (\frac{p_p}{3})^3 > 0 ,$$

$$p_p = -\frac{1}{48}[16 - 188J + 321J^2 - 188J^3 + 16J^4 \pm 12(1+J)J^{\frac{3}{2}}] , \tag{48}$$

$$\begin{aligned} q_p \; = \; & \pm\frac{1}{96}(44 - 65J + 44J^2)(1+J)J^{\frac{3}{2}} \\ - \; & \frac{1}{864}(800 - 2760J + 4116J^2 - 4277J^3 \\ + \; & 4116J^4 - 2760J^5 + 800J^6) . \end{aligned} \tag{49}$$

Similarly, the upper and lower signs ($\pm$) correspond to A_{p1} and A_{p2} respectively. From these results, we obtain the stable window width

$$W_3 = A_s(J) - \frac{1}{4}(7 - 10J + 7J^2) \tag{50}$$

and the plateau width of Lyapunov exponent

$$P_3 = A_{p,2}(J) - A_{p,1}(J) . \tag{51}$$

Hence the ratio of plateau width to stable window width of period-3 orbit

$$\alpha_3(J) = \frac{P_3(J)}{W_3(J)} .$$

In Fig. 3 we show the results of the above analytical calculation of the width ratios α_1, α_2, α_3, for period-1, 2, and 3 orbits of Henon map (27) as function of J.

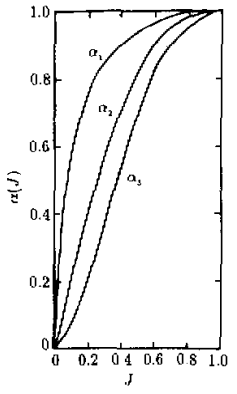

Fig. 3. *The width ratio α as function of J for period-1, 2, 3 orbits of Henon map.*

For higher order periodic orbits, we can obtain the stable window of attractor orbit and rotation region of eigenvalue using the same method. The general form of the discriminant function $F_N(J, A)$ for Henon map has been given in Ref. [7].

IV. THE STRUCTURE INVARIANCE OF LYAPUNOV EXPONENT PLATEAU FOR PERIOD DOUBLING SEQUENCE

The Jacobian value representing effective dissipation for a period-N orbit should be

$$J_{\text{eff}} = J^N \ . \tag{52}$$

If taking J_{eff} as the variable, we notice that from Eqs. (33) and (37)

$$\alpha_1(J_{\text{eff}}) = \alpha_2(J_{\text{eff}}) = \frac{2J_{\text{eff}}^{1/2}}{1 + J_{\text{eff}}} \ . \tag{53}$$

In other words, for period-1 and period-2 orbits of Henon map, the width ratios α_1 and α_2 as the function of effective Jacobian parameter are strictly the same. Is it only an event of coincidence? We make a conjecture that for all the period doubling orbits, there exist invariant property of Lyapunov exponent plateau structure, that is, the width ratio α as function of J_{eff} are exactly the same.

For proving this conjecture, we have to calculate numerically with high precision the stable window and Lyapunov exponent plateau of period-4 orbit which is generated via period doubling bifurcation of period-2 orbit. The discriminant function for period-4 orbit is

$$F_4(J, A) = J^2 - 2J(y_1y_2 + y_2y_3 + y_3y_4 + y_4y_1) + 8y_1y_2y_3y_4 \ . \tag{54}$$

In order to obtain stable window from $|F_4| \leq \frac{1}{2}(1+J^4)$ and plateau region from $|F_4| \leq J^2$) for period-4 orbit, $y_i (i = 1, 2, 3, 4)$ as the function of (J, A) should be given. By simply nu-

merical iterating of the map, we can only obtain attractor in the stable region. However, it is not possible to determine the two boundary points of this stable window with high precision for period-4 orbit. The reason is period doubling bifurcation at end point of stable window that will cause critical slowing down near the bifurcation point and reduce convergence of numerical iterating. We have to use Biham-Wenzel algorithm to solve unstable periodic orbit[8] by which we obtain the stable corridor and Lyapunov exponent plateau region in $J-A$ parameter plane $-1 \leq J \leq 1$ for period-2 and period-4 orbits as shown in Fig. 4(a). An enlargement of the narrow part of the corridor for stable and plateau region is shown in Fig. 4(b), (c).

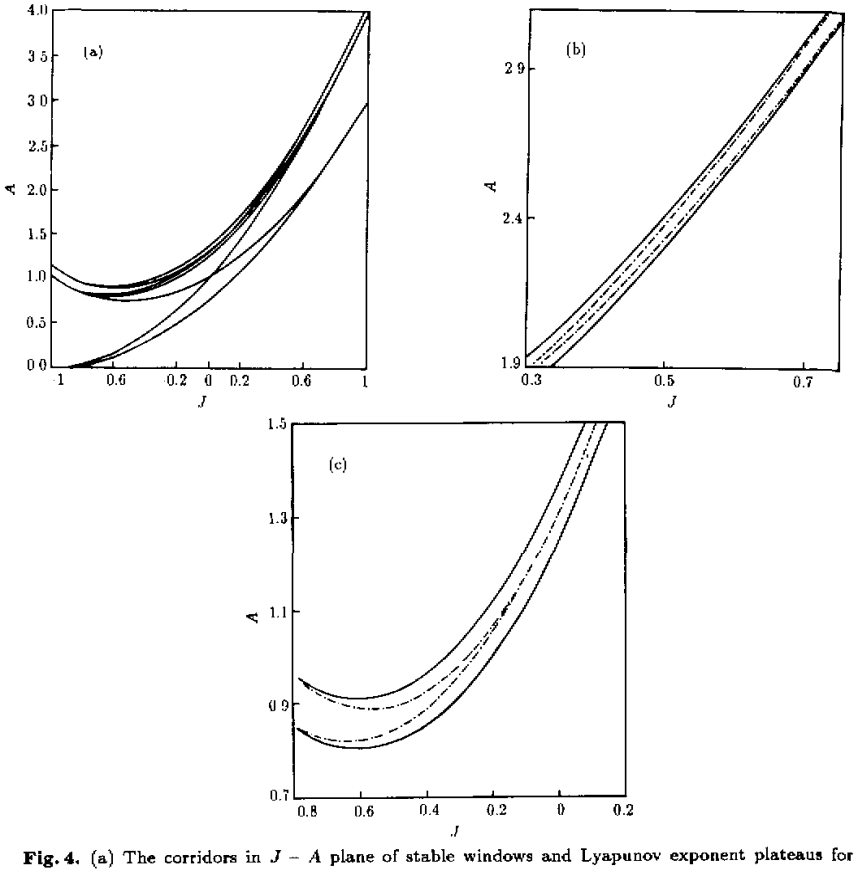

Fig. 4. (a) The corridors in $J - A$ plane of stable windows and Lyapunov exponent plateaus for period 1, 2, 4 orbits of Henon map. The end lines of plateau region cross each other at $J = 0$. (b), (c): An enlargement of the narrow part of the corridor for stable window and plateau region. Dashed lines represent the low and high end lines of plateau region.

In Fig. 5 we show the ratios α_1, α_2, α_4, of Lyapunov exponent plateau width to stable window width as the function of $J_{\text{eff}} = J^4$ for period-1, 2, 4 orbits of Henon map that belong to period doubling sequence. These function curves are the results obtained by the above numerical method. It is surprising that these curves coincide perfectly.

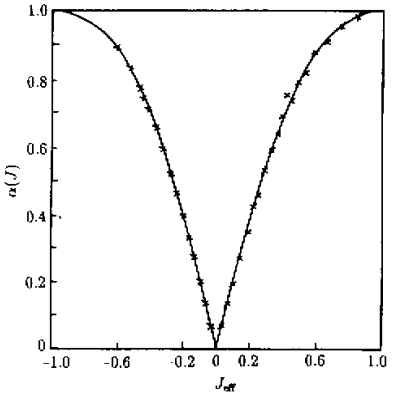

Fig. 5. The width ratio α as the function of $J_{\text{eff}} = J^N$ for period $N = 1, 2, 4$ orbits of Henon map. The small dots represent period-1, 2 orbits. The crosses "×" represent period-4 orbit.

Therefore, at least for period-1, 2, 4 orbits of Henon map, we have proven the correctness for the above conjecture about plateau structure invariance of Lyapunov exponents. Based on these results, the conjecture may be further generalized to infer that the above transition behavior of Lyapunov exponent is universal for all the orbits belonging to a period doubling sequence. That is, for any orbit with period $N = m \times 2^n$ $(n = 0, 1, 2, \cdots)$, the ratio of plateau width to stable region width $\alpha_N = P_N/W_N$ as the function of $J_{\text{eff}}^{(N)} = J^N$, namely $\alpha_N(J_{\text{eff}}^{(N)})$, is a universal function which remains invariant with increasing N.

Particularly, for $m = 1$, namely, for period-2^n $(n = 0, 1, 2, \cdots)$ orbits, the universal function is

$$\alpha_N(J_{\text{eff}}) = \frac{2|J_{\text{eff}}|^{1/2}}{1 + |J_{\text{eff}}|} . \tag{55}$$

For orbits with high period of Henon map, although it is not possible to determine analytically the trajectory solution and the end point values of plateau and stable window, the width ratio can be given by simple analytical functions

$$\alpha_4(J) = \frac{2J^2}{1 + J^4} , \quad \alpha_8(J) = \frac{2J^4}{1 + J^8} , \cdots$$

$$\alpha_{2^n}(J) = \frac{2J^{2^{n-1}}}{1 + J^{2^n}} . \tag{56}$$

Can the conjecture be proved true analytically? It is an interesting open question.

The crossover characteristic of plateau structure of Lyapunov exponent shown above is a reflection of new invariant dynamical property of orbits belonging to period doubling sequence. It reveals a new type of universal transition behavior between order and chaos in the nonlinear dynamical systems. The new invariance connected to period doubling bifurcation scenario is not only an asymptotic behavior as scaling invariance, but also a kind of invariance in dynamical stability transition followed strictly from the low period. The invariance appears as identical transition form of Lyapunov exponent plateau structure for all period doubling orbits. The further research and explanation of this new invariance will provide a deeper and richer understanding of basic universal law of transition between order and chaos in nonintegrable dynamical system changing from dissipative to conservative.

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