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ONE-DIMENSIONAL SANDPILE MODEL WITH STOCHASTIC SLIDE*

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A new kind of theoretical one-dimensional sandpile model is proposed. In contrast to the models studied previously, the sliding process in this model is assumed to be of stochastic nature. Numerical simulations show that the behavior of this model is apparently closer to the reality of true sandpile than the models considered previously. The universality and scaling of this model is also discussed.

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I. INTRODUCTION

In recent investigation of dissipative dynamical systems, Bak, Tang and Wiesenfeld[1] introduced the notion of self-organized criticality. They showed that such dynamical systems evolve naturally into a critical state through a self-organization process. This state is barely stable and, when it is perturbed, the resulting relaxation processes are scale invariant. These relaxation processes occur on all length and time scale up to limits determined by the finite size of the system. They suggested that long-range temporal correlation with a $1/f$ power spectrum could be understood in terms of self-organized criticality and that there is connection betwen $1/f$ noise and the spatial self-similar structure of the critical state.

The theoretical sandpile is a system studied extensively as a paradigm for self-organized criticality. All models investigated by several authors involve two kinds of steps[1-4]. First, a particle is added to a random site. Second, after the addition, a set of cellular automaton rules are applied, which give an algorithm for determining whether the system is still stable and, if not, how the pile rearrange itself. These rules are applied repeatedly until the avalanche ceases to occur. Criteria which determine what is stable and the rules for how an unstable pile will reorganize itself until it regains stability could differ from one another. However, in the models of Refs.[1-4] there is a common feature: the particle is added at a random site while the slide is deterministic.

By analogy with traditional critical phenomena, it was argued that near the critical

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state there exists a universal behavior and that the system can be characterized by several critical indices. Denoting by s the toppling size of sandpile, and δ the number of grains that drop off the edge when a particle is added, one finds in a large class of sandpile that the distribution functions $D(s)$ and $F(\delta)$ obey scaling laws

$$D(s) \propto s^{-\alpha} \tag{1}$$

and

$$F(\delta) \propto \delta^{-\beta}, \tag{2}$$

with s or δ varying in a quite wide range. Here α and β are constants, and both $D(s)$ and $F(\delta)$ are normalized to the total number of events of adding particles. With the finite size effect taken into account, these two distribution functions are fitted by using the finite-size scaling method or multifractal analysis method by some authors[1, 2].

Recently, experiments on real sandpiles have been made by some authors[5,6]. In the experiment in Ref. [5], the sandpile does not show any critical behavior. The avalanche of the sandpile is led to by increasing the slope of the sandpile. It differs from the theoretical study of sandpile where the avalanche is caused by adding particles to a random site in the sandpile. However, the results for the experiment in Ref.[6] are quite different. In this experiment, the total mass of the sandpile is recorded, and the distribution function $F(\delta)$ is obtained for various sizes of bases. For the sandpile with a small linear size L of the base, the scaling law (2) holds. For the system with a large L, however, it behaves like a relaxation oscillator, and the distribution function $F(\delta)$ becomes sharply peaked at nonzero values of δ and the scaling law (2) breaks down. Therefore, the criticality breaks down in the limit of large sandpile, in contrast to the theoretical models in Refs.[1 – 4]. According to their experiment, the authors of Ref. [6] believed that the presence or absence of self-organized criticality was related to a damping length scale which had not been fully understood. As far as we know, the power law (1) has not been checked experimentally yet, and the damping length scale has not been found in theoretical models yet.

The theoretical models studied previously mostly have deterministic slide: as long as the slope (or height) at every site of the sandpile is known the avalanche will lead the sandpile to a unique stable state. However, a recent study shows that in real sandpiles the behavior is different[7]. Since the grains may have different shape and size, the translational symmetry might be broken. On the basis of the stochastic nature of real sliding processes, we propose a new theoretical model. In this paper we will discuss the case of one-dimensional sandpile only, while a generalization to two-dimensional case will be considered in another paper.

The present model differs from the previous ones in two aspects. First, the region over which the grains are added is confined to a certain part of the sandpile, and this is in agreement with the experimental arrangement[6]. Second, the slide is stochastic. To characterize this feature, the threshold value at a site is allowed to change when this site accepts grains sliding from their neighbors. This difference makes the behavior of the model quite different from those with deterministic slide[1, 2], and a numerical simulation shows that this model is much closer to the reality of an actual sandpile than the previous ones.

Upon understanding this simple system we might gain an insight into the processes in real sandpiles.

II. THE MODEL

The basic variable $h(i)$ in the model proposed here is the height of the sandpile at the site i, where $0<i\leqslant L$, with L the base size of the sandpile. The local slope $z(i)$ of the pile is defined as

$$\begin{aligned} &z(i)=h(i)-h(i+1) \quad \text{for} \quad 0<i<L, \\ &z(L)=h(L). \end{aligned} \tag{3}$$

The boundary conditions are such that the grains can flow out of the system from the right side only. This corresponds to a closed boundary at the left edge $i=1$ and an open boundary at the right edge $i=L$. The process of adding a particle at the site i described by

$$h(i) \to h(i)+1 \quad \text{for} \quad 0<i\leqslant cL$$

or

$$z(i) \to z(i)+1, \quad z(i-1) \to z(i-1)-1, \tag{4}$$

with $0<c\leqslant 1$. When the slope at some site exceeds a threshold value Z_c

$$z(i)>Z_c, \tag{5}$$

this site is unstable and will give $s=z(i)-Z_c$ grains to the nearest site on the right,

$$\begin{aligned} &h(i) \to h(i)-s, \\ &h(i+1) \to h(i+1)+s \quad \text{for} \quad z(i)>Z_c \end{aligned}$$

or

$$\begin{aligned} &z(i) \to z(i)-2s, \\ &z(i\pm 1) \to z(i\pm 1)+s \quad \text{for} \quad z(i)>Z_c \end{aligned} \tag{6}$$

Closed and open boundary conditions are used for the left and right boundaries, respectively,

$$z(0)=0, \quad z(L) \to z(L)-s, \quad z(L-1) \to z(L-1)+s \quad \text{for} \quad z(L)>Z_c. \tag{7}$$

In order to introduce a stochastic slide we assume that the threshold at site i could change according to the number of grains falling from its left site $(i-1)$. In other words, when some particles slide from the left site, the threshold at this site may become lower, and the particles at this site might be easier to topple. Let $p_s(r)$ be the probability that the threshold Z_c become Z_c-r when the number of particles sliding from its left site is s. Note that there is considerable freedom in the choice of a form for the function $p_s(r)$. In general, we know that $p_s(r)\geqslant 0$, $p_0(r)=\delta_{r0}$. Assuming that the more grains fall the lower the threshold will be, then the average of r, i.e. $\langle r\rangle=\sum_{r=1}^{\infty} rp_s(r)$, must be a bounded and monotonically increasing function of s. In this paper we choose the following simple form for the function $p_s(r)$ for $s>0$:

$$p_s(r)=\begin{cases} 0 & \text{for } r<0 \text{ or } r>r_c; \\ 1-\lambda & \text{for } r=0; \\ \lambda/s & \text{for } 1\leqslant r\leqslant r_c-1; \\ (s-r_c+1)\lambda/s & \text{for } r=r_c<s+1, \end{cases} \tag{8}$$

with two parameters λ and r_c. The choice of the two parameters has great freedom. When $\lambda \to 0$, this model comes back to a trivial one-dimensional model[1], which has a minimally stable state, and has no self-organized criticality.

III. NUMERICAL RESULTS AND ANALYSIS

In this section, we consider the model with $c=0.5$, $\lambda=0.2$, $r_c=5$. Because of the stochastic nature of the slide, the behavior of this model differs in many respects from those of the previous models. If the time unit is defined as the interval of particle addition on the sandpile, we find that the average interval $\langle \Delta T_\delta \rangle$ between the successive events of the mass flow off the sandpile increases with the size of base L, as shown in Fig.1. We know that for a given sandpile the avalanche size δ cannot exceed $\frac{1}{2}r_c L^2$. Hence, if $\langle \Delta T_\delta \rangle$ increases with L in a power law, i.e., $\langle \Delta T_\delta \rangle \propto L^\gamma$, the exponent γ cannot be larger than 2, for $L \to \infty$. However, at some finite size L the slope

$$\gamma=\frac{\mathrm{d}\log\langle \Delta T_\delta \rangle}{\mathrm{d}\log L} \tag{9}$$

could be larger than 2.

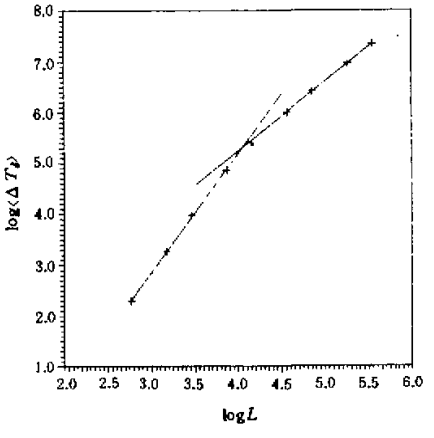

Fig.1. The averaged time interval $\langle \Delta T_\delta \rangle$ between two successive events of mass flow off the sandpile increases very rapidly as the base size of sandpile L increases. There is a turning point at about $L=L_c=64$.

Figure 1 shows that there exists a turning point. The turning point in this curve is at about $L_c=64$, which we call the damping length scale. This point divides the curve into two segments with different slopes. When $16\leqslant L\leqslant 64$, the slope $\gamma=2.34$, which is larger than 2. While $L>64$, the slope ($\gamma=1.37$), is smaller than 2. The scale of the avalanche for very small sandpile with $L<16$ is strongly confined by its size of the base, and will not be considered here. The existence of the damping length scale is due to the stochastic nature of the slide. In the beginning of an avalanche caused by adding a single grain, the threshold must be Z_c. With every step in the toppling process there is

some chance to lower the threshold. The larger the size L of the sandpile, the more events for decreasing the threshold will occur, and the lower the threshold $Z_c - r$ will be generated. This will certainly make the size of the avalanche larger, and result in a larger time interval ΔT_δ. Hence the value of γ could be larger for $L < L_c$. For even larger size of the sandpile, $L > L_c$, the small slope γ may be explained by the fact that the lowest threshold is already achieved. In the next section we will discuss the relations between the value of the damping scale and the value of the parameter λ. We find that larger values of λ will make the scaling length L_c smaller. Since the slide in a real sandpile is of a stochastic nature, we expect that the qualitative characteristics of the curve in Fig.1 could be verified experimentally. The experiment in Ref.[6] seems to show that the relation between $\langle \Delta T_\delta \rangle$ and L predicted in this model is qualitatively correct.

Now, we examine the universality and scaling of the sandpile in the present model. We find that very different sizes of avalanches could occur in the sandpile system with limited sizes. Numerical simulation shows that the distribution function $D(s)$ for a given size of the sandpile, with s being the toppling size of the sandpile, follows a power relation (1) for small s. We can obtain a group of distribution functions $D(s, L)$ when we change the base size L of the sandpile. Taking the finite size effect into account, we have tried a multifractal fit to these distribution functions $D(s, L)$

$$\frac{\log D(s, L)}{\log(L/L_0)} = g_s\left(\frac{\log(s/s_0)}{\log(L/L_0)}\right). \tag{10}$$

Here $g_s(x)$ is called a scaling function. With $s_0 = 2.0$ and $1/L_0 = 1.2$, these distribution functions can be fitted very well. The results are shown in Fig.2. This relation is similar to the previous model with deterministic slide[2].

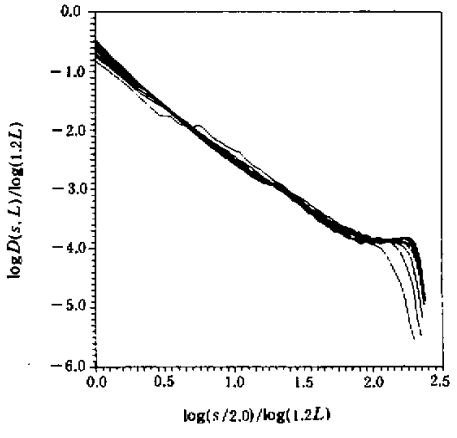

Fig. 2. A multifractal analysis fit for the distribution functions $D(s, L)$ with $L = 16, 24, 32, 48, 64, 96, 128, 192, 256$. The fitting parameters here are $s_0 = 2.0$ and $1/L_0 = 1.2$.

However, when we examine the distribution functions $F(\delta, L)$ of different sizes of sandpile, with L the base size of sandpile, and δ the total number of grains that drop off the edge when a grain is added, we find that the scaling behavior of these distribution functions $F(\delta, L)$ are different from the theoretical models discussed previously[2]. The numerical simulation results for $F(\delta, L)$ are shown in Fig. 3(a).

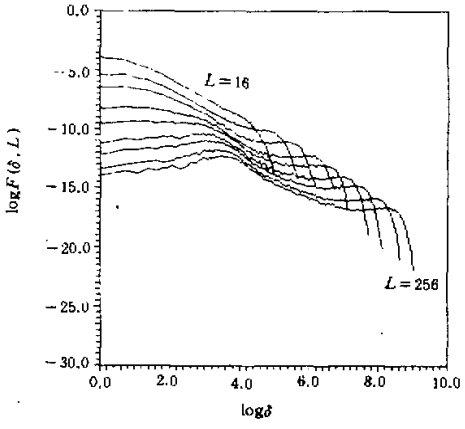

Fig. 3(a). The distribution functions $F(\delta, L)$ for $L=16$, 24, 32, 48, 64, 96, 128, 192, 256.

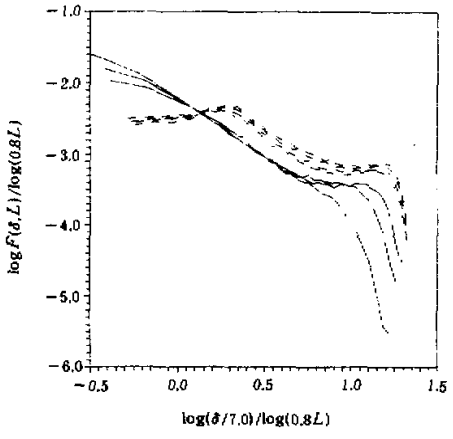

Fig. 3(b). The solid lines indicate the scaled $F(\delta, L)$ for $L=16-32$. For comparison the scaled $F(\delta, L)$ for $L=96-256$ are also shown by dashed lines. The fitting parameters in this figure are $\delta_0=7.0$ and $1/L_0=0.8$.

The fitting parameters in this figure are $\delta_0=2.0$ and $1/L_0=2.0$. Neither finite-size-scaling nor multifractal analysis can fit these distribution functions $F(\delta, L)$ for all sizes of sandpiles uniformly. However, if one tries to fit these functions $F(\delta, L)$ by

$$\frac{\log F(\delta, L)}{\log(L/L_0)} = g_s\left(\frac{\log(\delta,\delta_0)}{\log(L/L_0)}\right) \tag{11}$$

with $\delta_0=7.0$ and $1/L_0=0.8$, the result shown in Fig. 3(b) would be obtained. From this figure we can see that the functions for $16<L<L_c$ can be fitted together. On the other hand, with $\delta_0=2.0$ and $1/L_0=2.0$, these functions $F(\delta, L)$ for $L>L_c$ can be also fitted together, as shown in Fig. 3(c).

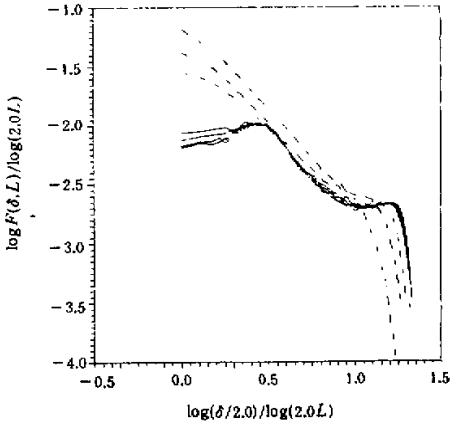

Fig. 3(c). The solid lines indicate the scaled $F(\delta, L)$ for $L=96-256$. For comparison, the scaled $F(\delta, L)$ for $L=16-48$ are also shown here by dashed lines. The fitting parameters in this figure are $\delta_0=2.0$ and $1/L_0=2.0$.

Hence the sandpiles with different sizes can be divided into two groups. The first group comprises sandpiles of small sizes with $L<L_c$. After rescaling, the relation (2) holds for a wide range of δ. The sandpiles with $L>L_c$ form the second group. Therefore, we believe that the sandpiles with different base size L lie in two different universality classes in our model. Because there exists the damping length scale in our model, and the value of γ is considerably large for $L<L_c$ and small for $L>L_c$, the behavior of this model is apparently closer to the experiment of real sandpiles[6]. In order to see this we consider the evolutions of the total mass M of the sandpile, and show this for $L=16$ in Fig. 4(a)–(c) and for $L=48$ in Fig. 4(d), respectively.

One could find that they are very similar to the experimental results in Ref. [6]. When $L=16$ a total number of 234 events of mass flow off the sandpile system have been recorded in 2400 events of addition of the particle. It is clear that the mass exhibits fluctuation

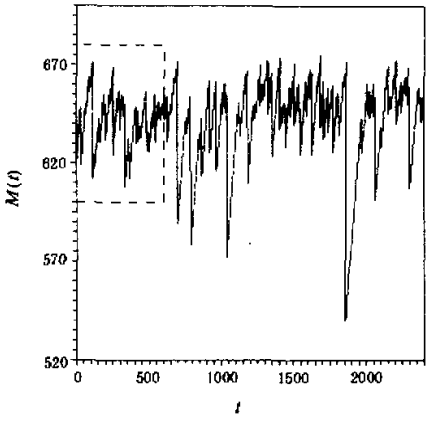

Fig. 4(a). Mass $M(t)$ of the sandpile with $L=16$ as a function of time t. Within 2400 events of addition of particles a total number of 234 events of the mass flow off the sandpile have been recorded.

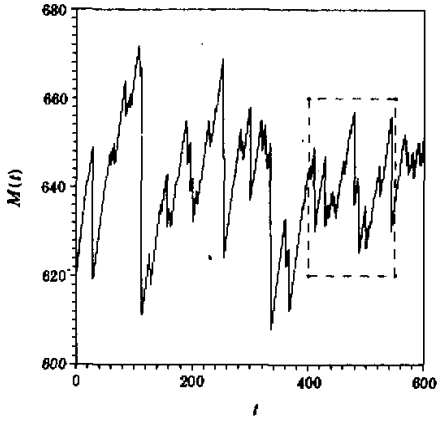

Fig. 4(b). magnification of the boxed region in 4(a).

over periods ranging from one to several hundred dropping events, with the sizes of the avalanche ranging between one and several hundred grains of sand. However, when $L=48$ there are only 15 events of avalanche in the same number of events of adding particle, and the fine-scale mass fluctuation, observed for the sandpile of $L=16$, has hardly

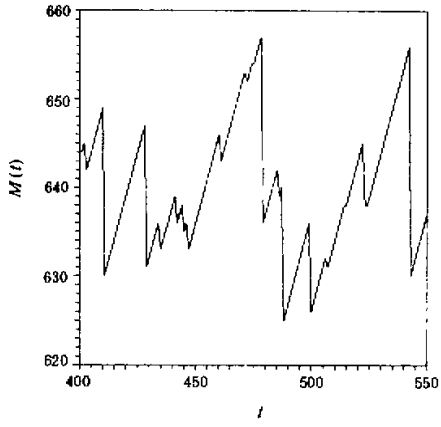

Fig. 4(c). Magnification of the boxed region in 4(b).

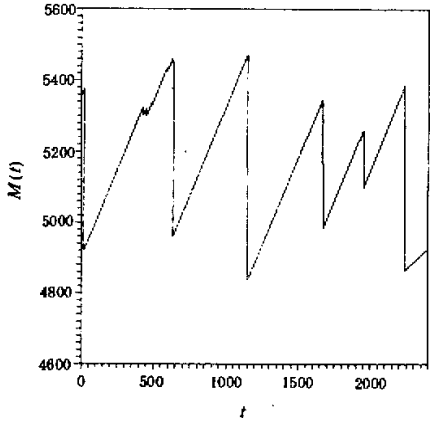

Fig. 4(d). Mass $M(t)$ for $L=48$, which behaves like a relaxation oscillator. There is only 15 events of mass flow off the sandpile in the same time interval. These behaviors are very similar to those measured in Ref.6.

been seen. Since the time interval ΔT_δ becomes considerably larger for large L one has to measure for longer time to get sufficient information about the evolution of M.

Although, as mentioned above, there exist many differences due to the different base sizes of sandpiles, the mass power spectra of the time series have the same $1/f^2$ form for

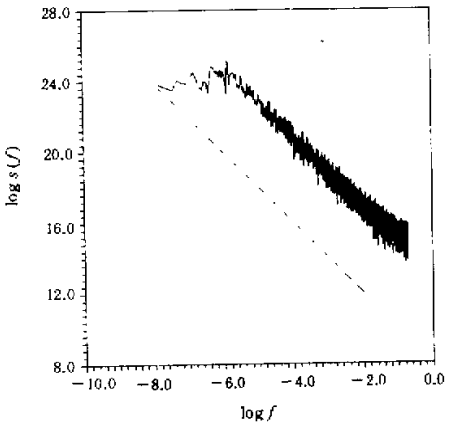

Fig. 5. The power spectrum of the mass fluctuations of the sandpile model with $L=32$. The dashed line (for comparison) shows a $1/f^2$ power spectrum.

different sizes of the sandpile. This is in agreement with the experiment in Ref.[6]. Figure 5 shows the power spectrum of the mass fluctuation of the sandpile with a base size $L=32$. This figure is obtained by averaging 100 samples, each of them contains a total of 131072 data about time evolution of M. The spacing between data points of the spectrum is about 7.63×10^{-6} step^{-1}, with the data at frequencies above 1.5×10^{-4} step^{-1} smoothed by averaging over 10^{-4} step^{-1} interval. The results are well described by $s(f)\sim f^{-2}$ when $f>2.5\times10^{-3}$ step^{-1}. This $1/f^2$ form spectrum is the same as that measured for a random walk. It indicates that the events of mass flow off the sandpile are not correlated to each other, in agreement with the theoretical analysis in Ref.[8].

IV. DISCUSSION

In the above section, we have given the numerical simulation results at the values of parameters $c=0.5$, $\lambda=0.2$, $r_c=5$ for model (8). The most important one among these results is that there exists a damping length scale. We are then interested in the parameter dependence of the damping length scale. Numerical simulation results of these relations are shown in Fig.6(a)−(c).

Firstly, we can see from Fig.6(a) that the damping length scale L_c decreases as λ increases, with the values of r_c and c fixed, although the curves of $\langle\Delta T_\delta\rangle$ versus L with different values of the parameter λ are almost parallel to each other. This result can easily be understood, as the larger the value of λ, the easier the decrease of the threshold, and the probability of lowering the threshold will then increase. Therefore, along the route of

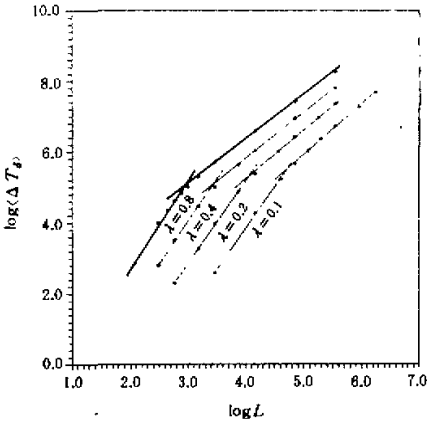

Fig. 6(a). The damping length scale L_c decreases as the value λ increases. Here $r_c=5$, $c=0.5$, and λ are 0.1, 0.2, 0.4, 0.8, respectively.

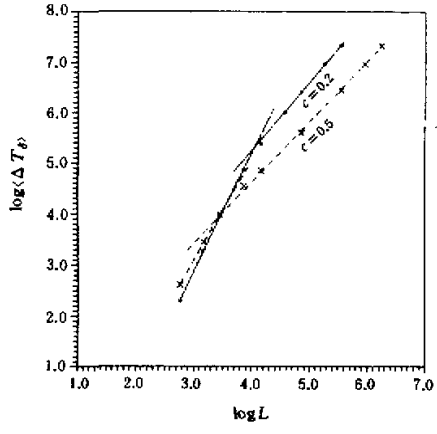

Fig. 6(b). The damping length scale L_c, which increases as the value c increases from 0.2 to 0.5. Here $\lambda=0.2$, $r_c=5$.

sliding events, the sites with decreased threshold become much more than for small λ. Hence the threshold may reach the lowest value even for smaller sizes of sandpiles. That means a small damping length scale L_c.

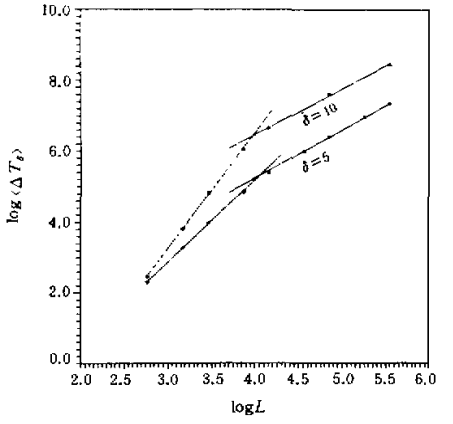

Fig. 6(c). The value of the slope for the part $L<L_c$, which increases when the value r_c increases from 5 to 10. Here $\lambda=0.2$, $c=0.2$.

Secondly, with fixed values of λ and r_c, the damping length scale L_c will become larger when the value of c increases, as shown in Fig. 6(b). The increase of the value c means that the region for the addition of grains becomes larger. So the grains could be added closer to the open edge of the sandpile, and the events of lowering the threshold induced by these grains would have a smaller probability. Therefore, the damping length scale L_c increases with the value of c.

Finally, the avalanche size δ is also closely related to the value of r_c, and, the larger the value of r_c, the more the sites where the threshold has a smaller value. Figure 6(c) shows these relations. Furthermore, we find that the slope of the curve $\langle \Delta T_\delta \rangle$ versus L for $L<L_c$ becomes larger when r_c increases.

In general, the change of these parameters can alter the properties of the system quantitatively, but not qualitatively. In other words, the damping length scale always exists.

The form of the function $p_s(r)$ has also influence upon the numerical simulation results. In our numerical simulation, a slightly more complicated function is also used for $p_s(r)$

$$p_s(r)=\begin{cases}(1-u)u^r & \text{for } r<r_c;\\ u^{r_c} & \text{for } r=r_c;\\ 0 & \text{for } r>r_c,\end{cases} \tag{12}$$

where $u=\exp(-1/s\lambda)$. Figure 7 shows the relations between the average interval $\langle \Delta T_\delta \rangle$ and the base size of the sandpile L, with $c=0.5$, $\lambda=0.2$, $r_c=5$ for this model and the model (8). From this figure we can see that the damping length scale still exists for the

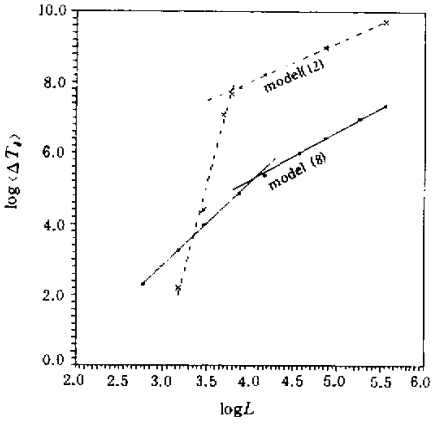

Fig. 7. The value of slope for the part of $L<L_c$ in model (12), which is larger than that in model (8).

choice (12) of the function $p_s(r)$, but the value of the slope in the part of $L<L_c$ for model (12) is larger than that for modal (8). The numerical simulation also shows that other properties of this model are qualitatively the same as the model (8).

V. CONCLUSION

In this paper, we have introduced a new kind of theoretical sandpile model. The main difference from previous models in mechanism is that the toppling process is stochastic instead of deterministic. This difference makes the behavior of the model change in many aspects, mostly closer to the real sandpile experiment. One important character of these behaviors is that there exists a damping length scale, in agreement with the sandpile experiment[6]. This character, however, does not exist in any one of the previous models. Besides this, the universality and the scaling of the system in our model is also different from those in previous models. Because the average time interval $\langle \Delta T_\delta \rangle$ increases rapidly with the size of the sandpile L, a more time consuming experiment is expected for verifying the critical behavior for larger sizes of sandpiles.

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