Time-like geodesic structure of a spherically symmetric black hole in the brane-world*

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Recently Malihe Heydari-Fard obtained a spherically symmetric exterior black hole solution in the brane-world scenario, which can be used to explain the galaxy rotation curves without postulating dark matter. By analysing the particle effective potential, we have investigated the time-like geodesic structure of the spherically symmetric black hole in the brane-world. We mainly take account of how the cosmological constant α and the stellar pressure β affect the time-like geodesic structure of the black hole. We find that the radial particle falls to the singularity from a finite distance or plunges into the singularity, depending on its initial conditions. But the non-radial time-like geodesic structure is more complex than the radial case. We find that the particle moves on the bound orbit or stable (unstable) circle orbit or plunges into the singularity, or reflects to infinity, depending on its energy and initial conditions. By comparing the particle effective potential curves for different values of the stellar pressure β and the cosmological constant α, we find that the stellar pressure parameter β does not affect the time-like geodesic structure of the black hole, but the cosmological constant α has an impact on its time-like geodesic structure.

Keywords: time-like geodesics, effective potential, spherically symmetric black hole

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1. Introduction

General relativity predicted many gravitational effects which were supported by subsequent observations. Lake[1] investigated light deflection in the Schwarzschild–de Sitter space-time. The planetary orbit of the test particle in a given space-time has been studied by many authors. By means of a detailed analysis of the corresponding effective potential, Jaklitsch *et al.*[2] investigated the time-like geodesic structure with a positive cosmological constant. Cruz *et al.*[3] studied the geodesic structure of the Schwarzschild–anti-de Sitter black hole. The analysis of the effective potential for null geodesics in the Reissner–Nordström–de Sitter and Kerr–de Sitter space-time was carried out in Refs. [4] and [5]. All possible geodesic motions in the extreme Schwarzschild–de Sitter space-time were investigated by Podolsky.[6] Chen and Wang[7−10] have investigated the orbital dynamics of a test particle in several gravitational fields with an electric dipole and a mass quadrupole, and in the extreme Reissner–Nordström black hole space-time. Exact solutions in closed analytic form for the geodesic motion in the Kottler space-time were considered by Kraniotis *et al.*[11] Kraniotis[12] investigated the geodesic motion of a massive particle in the Kerr and Kerr (anti)de Sitter gravitational field by solving the Hamilton–Jacobi partial differential equation. Other authors[13−15] have considered similar cases in recent years.

The brane-world scenario has developed our understanding of stellar structures and black holes. In the brane-world scenario, our familiar 4-dimensional (4D) space-time is a hypersurface (brane) in a 5-dimensional space-time (bulk)[16−21] for which all matter and gauge interactions reside on the brane, while gravity can propagate in the whole 5-dimensional space-time. Dadhich *et al.*[22] proposed the first static spherically symmetric exterior vacuum solution of the brane-world model. In Ref. [23] Germani and Maartens found an exact interior uniform density stellar solution in the brane-world. Consider-

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ing spherically symmetric solutions within the context of brane-world theory without mirror symmetry or any form of junction, Malihe and Sepangi[24] obtained the modified Tolman–Oppenheimer–Volkoff (TOV) interior solutions in two cases for a constant bulk. The first case is $c = 0$, and by using two of the field equations together with the energy–momentum conservation equation, they also obtained the exact exterior solution whose structure can be used to explain the galaxy rotation curves without postulating dark matter and without having to resort to modified Newtonian dynamics (MOND); for the case $c = \beta = 0$ and $\alpha \neq 0$, the solution represents a black hole in an asymptotically de Sitter space-time.

In this paper we concentrate on the spherically symmetric exterior solution which can be used to explain the galaxy rotation curves without assuming the existence of dark matter. By solving the Euler–Lagrange equations, we obtain the equation of motion, from which we define an effective potential containing two parameters. Here the parameter α is related to the cosmological constant and the parameter β represents the stellar pressure, respectively.

2. Spherically symmetric solution in the brane-world scenario

We first present a brief review of the spherically symmetric solution in the brane-world, which was obtained in Ref. [24]. The embedding of the brane-world in the bulk plays an essential role in the covariant formulation of the brane-world gravity, since it tells us how the Einstein–Hilbert dynamics of the bulk is transferred to the brane-world. The field equation for $g_{\mu\nu}$ with the confined matter represented by $T_{\mu\nu}$ is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \alpha^* \tau_{\mu\nu} - \lambda g_{\mu\nu} + Q_{\mu\nu}, \tag{1}$$

where

$$Q_{\mu\nu} = (K^{\rho}{}_{\mu} K_{\rho\nu}) - \frac{1}{2}(K_{\alpha\beta}K^{\alpha\beta} - K^2)g_{\mu\nu}. \tag{2}$$

The confined matter source on the brane is considered to be an isotropic perfect fluid

$$\tau_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \quad p = (\gamma - 1)\rho. \tag{3}$$

Choose the static spherically symmetric metric in the brane-world as

$$\begin{aligned} \mathrm{d}s^2 = & -\mathrm{e}^{\mu(r)}\mathrm{d}t^2 + \mathrm{e}^{\nu(r)}\mathrm{d}r^2 \\ & + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \end{aligned} \tag{4}$$

the York relation

$$K_{\mu\nu} = -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial \xi}, \tag{5}$$

and the conservation equations

$$\tau^{\mu\nu}{}_{;\nu} = 0. \tag{6}$$

For the static spherical symmetry metric (4), we have the following equations:

$$\begin{aligned} & \mathrm{e}^{-\nu}\bigg(-\frac{1}{r^2} + \frac{\nu'}{r}\bigg) + \frac{1}{r^2} \\ = & \ \alpha^*\rho + 3\alpha^2 + \frac{4\alpha\beta}{r} + \frac{\beta^2}{r^2}, \end{aligned} \tag{7}$$

$$\begin{aligned} & \mathrm{e}^{-\nu}\bigg(\frac{1}{r^2} + \frac{\mu'}{r}\bigg) - \frac{1}{r^2} \\ = & \ \alpha^*\rho - 3\alpha^2 - \frac{4\alpha\beta}{r} - \frac{\beta^2}{r^2}, \end{aligned} \tag{8}$$

$$p' + \frac{\mu'}{2}(\rho + p) = 0. \tag{9}$$

Solving Eqs. (7)–(9), we find that the exterior solutions are

$$\mathrm{e}^{\mu(r)} = \mathrm{e}^{-\nu(r)} = 1 - \frac{A_1}{r} - \alpha^2 r^2 - 2\alpha\beta\gamma - \beta^2. \tag{10}$$

This solution can be used to explain the galactic rotation curves without relying on the existence of dark matter and without assuming any new modified theory, e.g. MOND. The matching of the interior solution to that of the exterior then determines the integration constant as $A_1 = 2GM$.

3. Geodesic equation

The metric for the spherically symmetric space-time in the brane-world is

$$\begin{aligned} \mathrm{d}s^2 = & -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 \\ & + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \end{aligned} \tag{11}$$

where the lapse function $f(r)$ is

$$f(r) = 1 - \frac{2GM}{r} - \alpha^2 r^2 - 2\alpha\beta r - \beta^2. \tag{12}$$

In order to investigate the geodesic structure of the space-time described by Eqs. (11) and (12), we must solve the Euler–Lagrange equations for the variational problem associated with this metric. The corresponding Lagrangian is

$$\mathscr{L} = -f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2), \tag{13}$$

in which the dots represents the derivative with respect to the affine parameter τ along the geodesic. The equations of motion are

$$\dot{\Pi}_q - \frac{\partial \mathscr{L}}{\partial q} = 0, \tag{14}$$

where $\Pi_q = \partial \mathscr{L}/\partial \dot{q}$ is the momentum corresponding to coordinate q. Since the Lagrangian is independent of (t,ϕ), the corresponding conjugate momentum is conserved, therefore

$$\begin{aligned}\Pi_t &= -\left(1-\frac{2M}{r}-\alpha^2r^2-2\alpha\beta r-\beta^2\right)\dot{t}\\ &= -E, \qquad (15)\\ \Pi_\phi &= r^2\sin^2\theta\dot{\phi} = L, \qquad (16)\end{aligned}$$

where E and L are motion constants.

From the equation of motion for θ

$$\dot{\Pi}_\theta - \frac{\partial \mathscr{L}}{\partial \theta} = 0, \qquad (17)$$

we have

$$\frac{\mathrm{d}(r^2\dot{\theta})}{\mathrm{d}\tau} = r^2\sin\theta\cos\theta\dot{\phi}^2. \qquad (18)$$

We simplify the above equation by choosing the initial conditions $\theta=\pi/2$, $\dot{\theta}=0$, and $\ddot{\theta}=0$, equation (16) becomes

$$\Pi_\phi = r^2\dot{\phi} = L, \qquad (19)$$

and, from Eqs. (15) and (19), the Lagrangian (13) can be written in the following form:

$$\begin{aligned}2\mathscr{L} \equiv h &= \frac{E^2}{1-2M/r-\alpha^2r^2-2\alpha\beta r-\beta^2}\\ &\quad -\frac{\dot{r}^2}{1-2M/r-\alpha^2r^2-2\alpha\beta r-\beta^2}-\frac{L^2}{r^2}. \qquad (20)\end{aligned}$$

Now we solve the above equation for $\dot{r}^2$ in order to obtain the radial equation, which allows us to characterize possible moments of test particles and explicit solution of the equation of motion in the invariant plane

$$\begin{aligned}\dot{r}^2 &= E^2-\left(1-\frac{2M}{r}-\alpha^2r^2-2\alpha\beta r-\beta^2\right)\\ &\quad\times\left(h+\frac{L^2}{r^2}\right), \qquad (21)\end{aligned}$$

it is useful to rewrite the above equation of motion as a one-dimensional problem

$$\dot{r}^2 = E^2 - V_{\rm eff}^2, \qquad (22)$$

where $V_{\rm eff}^2$ is defined as an effective potential

$$\begin{aligned}V_{\rm eff}^2 &= \left(1-\frac{2M}{r}-\alpha^2r^2-2\alpha\beta r-\beta^2\right)\\ &\quad\times\left(h+\frac{L^2}{r^2}\right). \qquad (23)\end{aligned}$$

4. Time-like geodesic structure

For the time-like geodesic $h=1$, the corresponding effective potential becomes

$$\begin{aligned}V_{\rm eff}^2 &= \left(1-\frac{2M}{r}-\alpha^2r^2-2\alpha\beta r-\beta^2\right)\\ &\quad\times\left(1+\frac{L^2}{r^2}\right). \qquad (24)\end{aligned}$$

The radial geodesic corresponds to the motion of a particle without angular momentum, i.e. $L=0$, its effective potential is

$$V_{\rm eff}^2 = 1-\frac{2M}{r}-\alpha^2r^2-2\alpha\beta r-\beta^2. \qquad (25)$$

Depending on the value of the constant E, the following orbits are allowed in Fig. 1.

I) If $E > V_{\rm eff}^{\rm max}$, the particle can fall from infinity to the singularity.

II) If $0 < E < V_{\rm eff}^{\rm max}$, the particle can fall from infinity to a minimum distance $r=r_E$ and then come back to infinity. That is to say, the particle is only deflected. The other case corresponds to a particle which moves to the other side of the potential barrier and plunges into the singularity.

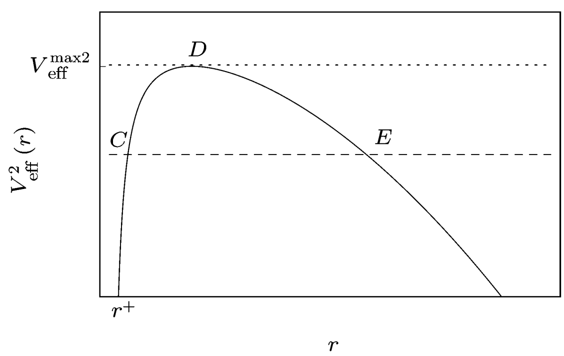

Fig. 1. The effective potential for radial particles for $h=1$, $L=0$, $\alpha=0.01$, $\beta=0.1$.

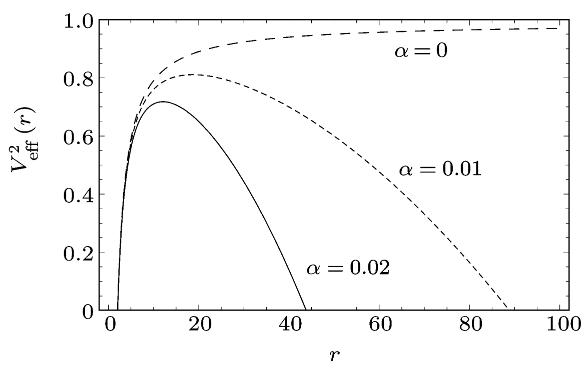

Fig. 2. The effective potential of radial particle for different values of the parameter α by fixed $h=1$, $L=0$, $\beta=0.1$.

Figure 2 shows how the parameter α affects the effective potential. We can see that when the value of α decreases, the peak of the effective potential rises, and if α becomes low enough, the particle can always fall from infinity to the singularity, with no possibility of going back.

The properties of the effective potential for parameter β are different from the case for α. In Fig. 3. we can see that the maximum value of $V_{\text{eff}}^{\max}$ becomes larger as the value of β increases, so the particle falls from infinity to a minimum distance $r = E$ and then comes back to infinity.

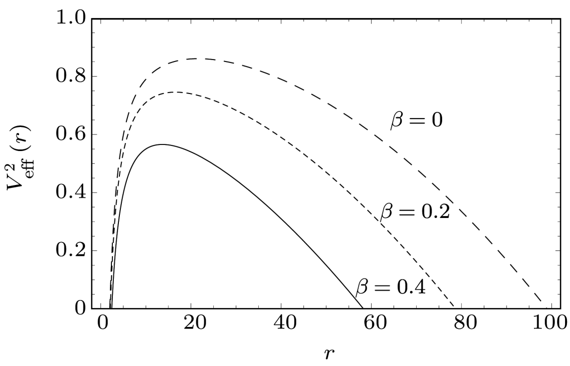

Fig. 3. The effective potential of a radial particle for different values of the parameter β with fixed $h = 1$, $L = 0$, $\alpha = 0.01$.

For a non-radial particle, the constant $L \neq 0$, and the corresponding radial equation takes the following form:

$$\begin{aligned} \dot{r}^2 &= E^2 - \left(1 - \frac{2M}{r} - \alpha^2 r^2 - 2\alpha\beta r - \beta^2\right) \\ &\quad \times \left(1 + \frac{L^2}{r^2}\right). \end{aligned} \tag{26}$$

We plot the effective potential of the non-radial particle in Fig. 4. The following orbits are allowed to depend on the value of the constant E.

(i) If $E^2 \leq {E_{\text{I}}}^2$, there are three possible cases. Firstly the particle will be on a stable circular orbit at $r = r_c$. Secondly the particle at the point B plunges into the singularity, and thirdly the particle at the point C goes back to infinity.

(ii) If ${E_{\text{I}}}^2 < E^2 < {E_{\text{II}}}^2$, the particle orbits on a bound orbit in the range $r_{\text{P}} < r < r_{\text{A}}$, where r_{P} and r_{A} are the perihelion and aphelion distance, respectively. The particle at the point D plunges into the singularity, and the particle at the point I goes back to infinity.

(iii) If ${E_{\text{II}}}^2 < E^2 < {E_{\text{M}}}^2$, there are two possible cases. First, the particle plunges into the singularity; second, the particle which moves to the other side of the potential barrier falls from infinity to a minimum distance $r = r_{\text{K}}$ and then goes back to infinity, i.e., the particle is only deflected.

(iv) If $E^2 > {E_{\text{M}}}^2$, the particle plunges into the singularity or flies to infinity, which is decided by its initial direction of motion.

(v) If $E^2 = {E_{\text{M}}}^2$ or $E^2 = {E_{\text{II}}}^2$, the particle can orbit on an unstable circular orbit at $r = r_{\text{F}}$ or $r = r_{\text{H}}$, respectively. For $r = r_{\text{F}}$, what is more likely is that a particle will orbit on a bound orbit in the range $r_{\text{F}} < r < r_{\text{J}}$, where r_{F} and r_{J} are the perihelion and aphelion distance, respectively, or just plunge into the singularity. For $r = r_{\text{H}}$, the particle will more likely either plunge into the singularity or fly to infinity.

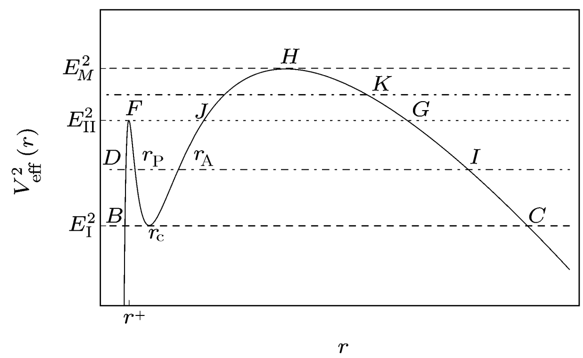

Fig. 4. The effective potential of a non-radial particle for $h = 1$, $L = 4$, $\alpha = 0.001$.

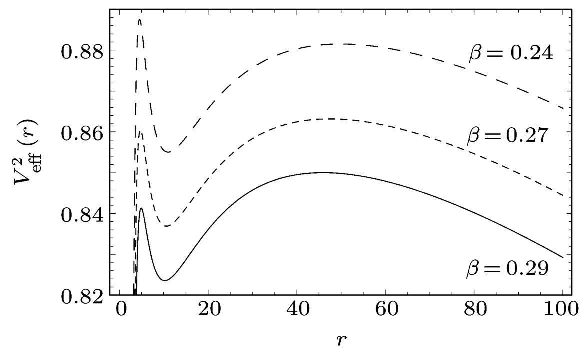

Fig. 5. The effective potential of a non-radial particle for different values of the parameter β with fixed $h = 1$, $L = 4$, $\alpha = 0.001$.

Furthermore we consider the effects of the parameters α and β on the effective potential curves in Figs. 5 and 6. From Figs. 5 and 3, we can see that the β does not change the shape of the effective potential curve, so we can conclude that the stellar pressure parameter β does not affect the time-like geodesic structure of the black hole, but only impacts on the level of the effective potential. However, from Figs. 6 and 2, we find that the cosmological constant α impacts not only the shape of the effective potential curve, but also the time-like geodesic structure of the black hole.

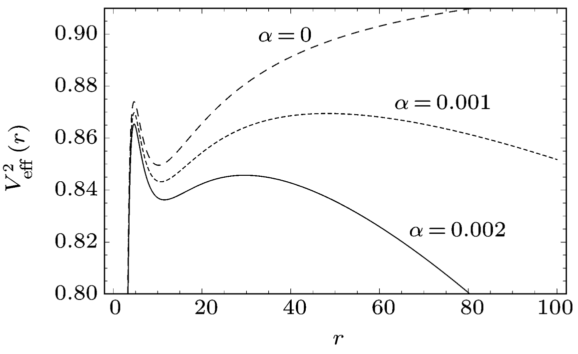

Fig. 6. The effective potential of a non-radial particle for different values of the parameter α with fixed $h = 1$, $L = 4$, $\beta = 0.26$.

5. Conclusions

By analysing the particle effective potential in spherically symmetric black hole in the brane-world, we have investigated its time-like geodesic structure. We have obtained the following results. i) For radial time-like geodesics, there are two cases: a) if the energy of the particle is higher than the peak value of the effective potential, the particle only falls from infinity to the singularity; b) if the energy of the particle is lower than the peak value of the effective potential, it falls from infinity to a minimum distance and then comes back to infinity, or plunges into the singularity, depending on its initial conditions. ii) For non-radial time-like geodesics, the time-like geodesic structure is more complex than the radial case. We have found that the particle moves on the bound orbit or stable (unstable) circlar orbit, or plunges into the singularity, or flies to infinity, depending on its energy and initial conditions. By comparing the particle effective potential curves for different values of the stellar pressure parameter β and the cosmological constant α, we have found that β does not affect the time-like geodesic structure of the black hole, but α impacts on the shape of the effective potential, that is to say, it affects the time-like geodesic structure of the black hole.

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