Phase diagram of interacting fermionic two-leg ladder with pair hopping

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Phase diagram of interacting fermionic two-leg ladder with pair hopping*

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We study the phase diagram of the interacting fermionic two-leg ladder, which is featured by pair hopping and interactions of singlet and triplet superconducting channels. By using Abelian bosonization method, we obtain the full phase diagram of our model. The superconducting triplet pairing phase is characterized by a fractional edge spin and interpreted as two Kitaev chains under the mean filed approximation. The pair hopping will give rise to spin-density-wave (SDW) orders and can also support Majorana edge modes in spin channel. At half filling, the resulting Majorana-SDW phase shows additional fractionalization in charge channel, and can be interpreted as two Su–Schrieffer–Heeger (SSH) chains in the mean field regime.

Keywords: two-leg ladder, pair hopping, bosonization, Majorana fermion

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1. Introduction

Topologically ordered phases of matter have been a hot topic in condensed matter physics since the discovery of the integer quantum Hall effect.[1] As a new kind of order, topological order can exist in two-dimensional (2D) and higher dimensional gapped systems,[2,3] and even in one-dimensional (1D) fermionic gapless systems.[4] Topological order is related to the ground state degeneracy, which is shown to be invariant against weak perturbations[5,6] and determines the statistics of quasiparticles.[7,8] Generally, there exist protected gapless edge states in topological phases.[9–13] For the special 1D case, the gapless edge states are called soliton in dimerized Su–Schrieffer–Heeger (SSH) model proposed for polyacetylene[14] and Majorana fermions in Kitaev model for superconducting quantum wires.[15] Majorana fermions are of great interest due to their non-Abelian statistics and potential application in quantum computation.[16–19] The emergence of Majorana fermions has been extensively studied in theory[20–26] and experiment[27–29] in a number of superconducting systems.

In 1D interacting systems, Landau–Fermi liquid breaks down to Luttinger liquid. The interaction will drastically modify the system, making it different from the noninteracting one.[30–33] Recently, a series of progresses point out that number-conserving setting by mutually coupling 1D wires offers an alternative route to create Majorana edge states.[34–38] Besides, developments on experiment make it possible to probe and simulate these fascinating topological phases.[39] The coupled 1D wires can also be realized by trapping ultracold atoms in the optical lattices in a relatively controllable and flexible manner.[40] These systems always have parity symmetry $Z_2^{\rm f}$, which is $+1$ for even number of fermions and -1 for odd number of fermions. The topological nontrivial phases turn out to be protected by $Z_2^{\rm f}$ symmetry, and the edge states may be understood as a consequence of symmetry-protected topological (SPT) fractionalization.[41] To gain a further insight into these topological phases in one dimension, we want to throw light on how interactions drive ground states and their relations with the noninteracting situations.

In this paper, we consider the interacting fermionic two-leg ladder featured by pair hopping,[34,37] which is believed to be the key to realizing Majorana fermions in the number-conserving systems. Compared with the charge parity $Z_2^{\rm f}$ describing the total fermion number, there is a spin parity $P_{\rm s}$ symmetry that describes the relative charge between the two legs.[42] Apart from pair hopping, we also introduce interactions of superconducting pair of two channels.[4] Analytically, we show that this interacting model exhibits abundant topological phases and demonstrate its relation with the noninteracting case to some extent.

The outline of this paper is as follows. In Section 2, we introduce our model and obtain the effective Hamiltonian by using the standard bosonization method.[43,44] In Sections 3 and 4, we study the weak-coupling phase diagram of our model, which is composed of one trivial and two nontrivial phases. The topological properties of the two nontrivial phases are em-

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bodied in their edge excitations. To gain a further insight into the nontrivial phases and their relations with the noninteracting situation, we use the mean field theory to show their analogies with Kitaev chains or dimerised SSH chains in Section 5. Finally, we give some discussions and conclude with a summary.

2. Model and effective field theory

As the starting point and schematically shown in Fig. 1, we consider fermionic two-leg ladder described by the Hamiltonian $H = H_0 + H_{\rm s} + H_{\rm d} + H_{\rm w}$

$$
\begin{aligned}
H_0 &= -t\sum_{j\alpha}(c^\dagger_{j\alpha}c_{j+1\alpha} + \text{H.c.}) - \mu\sum_{j\alpha}c^\dagger_{j\alpha}c_{j\alpha},\\
H_{\rm s} &= g_{\rm s}\sum_j \Delta^\dagger_{{\rm s}j}\Delta_{{\rm s}j}, \quad H_{\rm d} = g_{\rm d}\sum_j \Delta^\dagger_{{\rm d}j}\Delta_{{\rm d}j},\\
H_w &= w\sum_j(c^\dagger_{j\uparrow}c^\dagger_{j+1\uparrow}c_{j\downarrow}c_{j+1\downarrow} + \text{H.c.}),
\end{aligned}
\tag{1}
$$

where $c_{j\alpha}$ is the annihilation operator of pseudospin α (leg index) on site j, $\Delta_{{\rm s}j} = {\rm i}c_{j\alpha}\sigma^y_{\alpha\beta}c_{j+1\beta}$ is the spin singlet pairing, and $\Delta_{{\rm d}j} = c_{j\alpha}\sigma^x_{\alpha\beta}c_{j+1\beta}$ is the z component of spin triplet pairing $c_j{\rm i}(\hat{d}^z\cdot\hat{\sigma})\sigma^y c_{j+1}$.[45] Specially, w is the strength of pair hopping, which can be generated by laser-assisted tunneling.[37] The Hamiltonian shows reflection symmetry R defined as $c_\uparrow \leftrightarrow c_\downarrow$, prohibiting single particle σ_y process $c^\dagger_\alpha\sigma^y_{\alpha\beta}c_\beta$; time-reversal symmetry T is defined as $c_\uparrow \to c_\downarrow$, $c_\downarrow \to -c_\uparrow$, prohibiting both σ_y and σ_x process $c^\dagger_\alpha\sigma^x_{\alpha\beta}c_\beta$. Finally, the global spin $SU(2)$ symmetry breaks down to a $\mathbb{Z}_2$ symmetry characterized by spin parity operator $P_{\rm s} = (-)^{S^z}$, because the total spin component S^z changes only by multiples of ± 2. The superconducting pairing strengths $g_{\rm s}$ and $g_{\rm d}$ drive the ground state in the opposite direction, whereas the pair hopping w term is the key to realizing Majorana edge states. For simplicity, possible Hubbard interactions that may modify the ground states are not considered in our study.

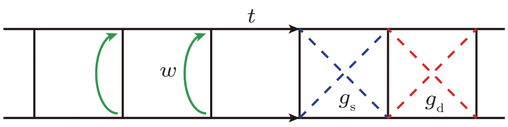

Fig. 1. Fermionic two-leg ladder with pair hopping (w) and interactions of singlet ($g_{\rm s}$) and triplet ($g_{\rm d}$) channels.

In the weak interaction regime, we take the method of Abelian bosonization to deal with these four-fermion terms. Naturally, we are interested in the equal density case $\bar{\rho}_\sigma = \rho$ with total charge as an even number. Linearizing spectrum of H_0 around two fermi points $k_{\rm F} = \pi\rho$, we decompose fermion operators into right and left movers $c_{j\sigma}/\sqrt{a} \to \psi_\sigma(x) = R_\sigma {\rm e}^{{\rm i}k_{\rm F}x} + L_\sigma {\rm e}^{-{\rm i}k_{\rm F}x}$. The right/left mover is expressed through bosonic fields $R_\sigma/L_\sigma = (1/\sqrt{2\pi a})F_{{\rm R/L},\sigma}{\rm e}^{{\rm i}\sqrt{\pi}(\pm\phi_\sigma - \theta_\sigma)}$, where a is the lattice spacing, $F_{{\rm R/L},\sigma}$ is the Klein factor that ensures anticommutation relation of fermi operators, and conjugated boson fields satisfy commutation relation $[\phi_\sigma(x), \theta_{\sigma'}(x')] = {\rm i}\delta_{\sigma\sigma'}\Theta(x-x')$. To see charge and spin degree of freedom more explicitly, we identify charge and spin boson $\phi_{\rm c/s} = (\phi_\uparrow \pm \phi_\downarrow)/\sqrt{2}$, $\theta_{\rm c/s} = (\theta_\uparrow \pm \theta_\downarrow)/\sqrt{2}$. Away from half filling, we finally arrive at the low energy effective Hamiltonian $H = \int {\rm d}x(\mathscr{H}_{\rm c} + \mathscr{H}_{\rm s})$, where

$$
\begin{aligned}
\mathscr{H}_{\rm c} &= \frac{v_{\rm c}}{2}[\frac{1}{K_{\rm c}}(\partial\phi_{\rm c})^2 + K_{\rm c}(\partial\theta_{\rm c})^2],\\
\mathscr{H}_{\rm s} &= \frac{v_{\rm s}}{2}[\frac{1}{K_{\rm s}}(\partial\phi_{\rm s})^2 + K_{\rm s}(\partial\theta_{\rm s})^2]\\
&\quad + g_\phi\cos(\sqrt{8\pi}\phi_{\rm s}) + g_\theta\cos(\sqrt{8\pi}\theta_{\rm s}),
\end{aligned}
\tag{2}
$$

with the charge and spin Luttinger parameters

$$
\begin{aligned}
K_{\rm c} &= \sqrt{\frac{\pi v_{\rm F}}{\pi v_{\rm F} + 4g_{\rm s}\cos^2 k_{\rm F} + 4g_{\rm d}\sin^2 k_{\rm F}}},\\
K_{\rm s} &= \sqrt{\frac{\pi v_{\rm F} + 4(g_{\rm d} - g_{\rm s})\sin^2 k_{\rm F}}{\pi v_{\rm F} - 4g_{\rm s}}},
\end{aligned}
\tag{3}
$$

where effective charge and spin interactions are $g_\phi = 2(g_{\rm s}\cos^2 k_{\rm F} - g_{\rm d}\sin^2 k_{\rm F})/\pi^2$, $g_\theta = -2w\sin^2 k_{\rm F}/\pi^2$. Spin and charge degrees of freedom are clearly separated and we do not show the explicit expression of charge/spin velocity $v_{\rm c/s}$ here. Situations at half filling is very subtle and we will discuss it later.

3. Superconducting phases

It is worthwhile to write down the bosonization forms of singlet and triplet pairing order parameters

$$
\begin{aligned}
O_{\rm ss} &\sim R_\uparrow L_\downarrow + L_\uparrow R_\downarrow \sim {\rm e}^{-{\rm i}\sqrt{2\pi}\theta_{\rm c}}\cos\sqrt{2\pi}\phi_{\rm s},\\
O_{\rm ts} &\sim R_\uparrow L_\downarrow - L_\uparrow R_\downarrow \sim {\rm e}^{-{\rm i}\sqrt{2\pi}\theta_{\rm c}}\sin\sqrt{2\pi}\phi_{\rm s},
\end{aligned}
\tag{4}
$$

and the spin-density wave (SDW) order parameters

$$
\begin{aligned}
O^x_{\rm sdw} &\sim R^\dagger_\uparrow L_\downarrow + R^\dagger_\downarrow L_\uparrow \sim {\rm e}^{-{\rm i}2k_F x}{\rm e}^{-{\rm i}\sqrt{2\pi}\phi_{\rm c}}\sin\sqrt{2\pi}\theta_{\rm s},\\
O^y_{\rm sdw} &\sim R^\dagger_\uparrow L_\downarrow - R^\dagger_\downarrow L_\uparrow \sim {\rm e}^{-{\rm i}2k_F x}{\rm e}^{-{\rm i}\sqrt{2\pi}\phi_{\rm c}}\cos\sqrt{2\pi}\theta_{\rm s}.
\end{aligned}
\tag{5}
$$

Here, we just consider these two types of order parameters, because they are the most possible ones emerging in the ground-state of our model. Actually, there is no true long range order in one dimension, whereas the local orders may emerge with power law decaying correlation functions (equal-time) in the Luttinger regime

$$
\langle TO(x)O^\dagger(0)\rangle \sim |x|^{-2\Delta},
\tag{6}
$$

where Δ is the scaling dimension of operator O, and $\Delta_{\rm ss/ts} = (K_{\rm c}^{-1} + K_{\rm s})/2$, $\Delta^{x(y)}_{\rm sdw} = (K_{\rm c} + K_{\rm s}^{-1})/2$.

Generally, the ground state is characterized by the order parameters whose correlation function decays the most slowly. Interestingly, we find that the critical points $K_{\rm c(s)} = 1$ arrive at the same condition $g_{\rm d}\tan^2 k_{\rm F} + g_{\rm s} = 0$. Apparently, negative $g_{\rm s}$ ($g_{\rm d}$) favors singlet (triplet) pairing. This superconducting pairing phase (see Fig. 2) is achieved if $K_s < 1$ and $K_c > 1$, which is satisfied when $g_d \tan^2 k_F + g_s < 0$. Moreover, negative g_ϕ pins $\sqrt{2\pi}\phi_s = n\pi$, resulting nonzero singlet order O_{ss}. In contrast, positive g_ϕ pins $\sqrt{2\pi}\phi_s = \pm\pi/2$, resulting in nonzero triplet order O_{ts}. The phase transition point $g_\phi = 0$ results in the phase boundary $g_d \tan^2 k_F - g_s = 0$, which shows a relationship with pairing strength $g_{s/d}$ and particle density ρ. Specially, there is a phase transition at quarter filling for the case $g_s = g_d$, where the pairing Hamiltonian becomes a simple form $H_s + H_d \sim -\sum_i n_{i\uparrow} n_{i+1\downarrow}$.

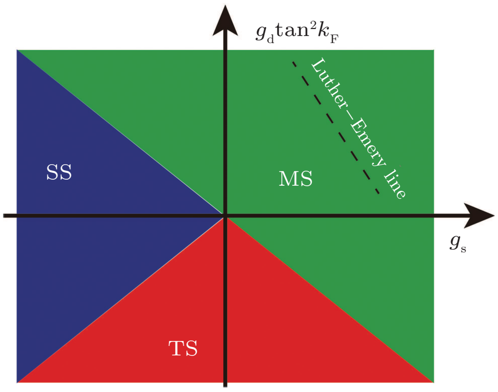

Fig. 2. Phase diagram of the weakly interacting fermionic two-leg ladder with pair-hopping. Superconducting singlet (SS) phase, superconducting triplet (TS) phase, and Majorana-SDW (MS) phase are obtained from bosonization description, with phase boundaries $g_d \tan^2 k_F \pm g_s = 0$. Along the Luther–Emery line, Majorana modes can be realized.

The singlet paring phase is trivial whereas the triplet paring phase is topological nontrivial. The topological properties of superconducting phases are present in the fractionalization at the ending points of two-leg ladder. Considering a system of length L with open boundary condition, there exists strong backscattering potential at both ends taking the form

$$-|v|\sum_\sigma (R_\sigma^\dagger L_\sigma + \text{H.c.}) = -\frac{2|v|}{\pi}\cos(\sqrt{2\pi}\phi_c)\cos(\sqrt{2\pi}\phi_s). \tag{7}$$

Such potential pins $\sqrt{2\pi}\phi_{c(s)} = n\pi$ at the boundaries. The triplet phase is characterized by $\sqrt{2\pi}\phi_s = n\pi + \pi/2$ in the bulk, resulting in a minimal kink $\pi/2$ of $\sqrt{2\pi}\phi_s$ at the ends. Such a kink of ϕ_s corresponds to an accumulation of edge spin

$$S_z = \int \frac{n_\uparrow - n_\downarrow}{2} \mathrm{d}x = \int \frac{\partial_x \phi_s}{\sqrt{2\pi}} \mathrm{d}x = \pm\frac{1}{4}, \tag{8}$$

half the spin of an electron.[4] The same fractional spin appears at the edge of a time-reversal invariant fully gapped topological superconductor.[46]

4. Majorana-SDW (MS) phases

The SDW orders are favored if g_θ term becomes relevant, which is the case when $g_d \tan^2 k_F + g_s > 0$. The ground state is characterized by $\theta_s = 0$, $\sqrt{\pi/2}$ for $w > 0$ and $\theta_s = \sqrt{\pi/8}$, $3\sqrt{\pi/8}$ for $w < 0$, with nonzero value of SDW order O^y_{sdw} and O^x_{sdw}, respectively. The ground state is two-fold degenerate in spin channel and within the topological $\mathbb{Z}_2$ phase. In bosonization description, spin parity operator P_s takes the form

$$P_s = \mathrm{e}^{\mathrm{i}\pi(N_\uparrow - N_\downarrow)/2} = \mathrm{e}^{\mathrm{i}\sqrt{\pi/2}\int \mathrm{d}x \partial_x \phi_s}. \tag{9}$$

It is easy to find $P_s = \pm 1$ and $P_s^\dagger \theta_s P_s = \theta_s + \sqrt{\pi/2}$. Since θ_s is periodic with $\sqrt{2\pi}$, one can define the state of even (odd) spin parity by symmetric (antisymmetric) combination $|\pm\rangle = |\theta_s = 0\rangle \pm |\theta_s = \sqrt{\pi/2}\rangle$ when $w > 0$, or $|\pm\rangle = |\theta_s = \sqrt{\pi/8}\rangle \pm |\theta_s = 3\sqrt{\pi/8}\rangle$ when $w < 0$, satisfying $P_s|\pm\rangle = \pm|\pm\rangle$.[4] We refer to this phase as Majorana-SDW, because it can also support Majorana edge modes in spin channel.

To show the Majorana edge modes, we refermionize Hamiltonian H_s at Luther–Emery point $K_s = 2$, where it is equivalent to free massive Dirac fermions. We rescale the bosonic fields $\tilde{\theta}_s = \sqrt{K_s}\theta_s$, $\tilde{\phi}_s = \phi_s/\sqrt{K_s}$, and introduce chiral fields $\tilde{R}_s/\tilde{L}_s = (1/\sqrt{2\pi})F_{R/L,s}\mathrm{e}^{\mathrm{i}\sqrt{\pi}(\pm\tilde{\phi}_s - \tilde{\theta}_s)}$, where $F_{R/L,s}$ is spinon Klein factor. Neglecting irrelevant terms, we rewrite $\mathscr{H}_s$ in terms of dirac fermions

$$\mathscr{H}_s = -\mathrm{i}v_s(\tilde{R}_s^\dagger \partial_x \tilde{R}_s - \tilde{L}_s^\dagger \partial_x \tilde{L}_s) - \mathrm{i}m(\tilde{R}_s^\dagger \tilde{L}_s^\dagger - \tilde{L}_s \tilde{R}_s), \tag{10}$$

with $m = 2w\sin^2 k_F/\pi$. This Hamiltonian corresponds to the continuum limit of 1D Kitaev model of p-wave superconductor,[15] which is known to supports Majorana zero modes at the boundaries

$$\gamma_1 \sim \int \mathrm{d}x(\tilde{R}_s + \tilde{L}_s^\dagger)\mathrm{e}^{-x/\xi}, \quad \gamma_2 \sim \mathrm{i}\int \mathrm{d}x(\tilde{R}_s - \tilde{L}_s^\dagger)\mathrm{e}^{x/\xi}, \tag{11}$$

where $\xi = v_s/m$ is the correlation length, measuring the localization of edge modes. Periodic boundary condition requires $\tilde{R}_s = \tilde{L}_s$ at the ends.[34] Using the two Majorana zero modes, one can define complex fermion operator $a = (\gamma_1 + \mathrm{i}\gamma_2)/2$, which satisfies the fermion anticommutation relation $\{a, a^\dagger\} = 1$. The degenerate ground states $|G_0\rangle$ and $|G_1\rangle = a^\dagger|G_0\rangle$ differ by one a fermion. Spin parity operator becomes $P_s = \mathrm{e}^{\mathrm{i}\pi\rho_s} = (-)^{a^\dagger a}$, opposite sign for $|G_0\rangle$ and $|G_1\rangle$. So we conclude the ground state $|G_{0,1}\rangle$ is exactly the states $|\pm\rangle$ mentioned above.

5. Mean field description

To understand the weak-coupling phase diagram of our model and its relation with the noninteracting situation, we use mean field theory to address the topological nature of ground states. Although mean field theory is not very suitable in 1D systems, it turns out to yield the correct Bogoliubov spectrum despite the fact that there is no true Bose–Einstein condensation.[47,48]

In topological triplet phase, H_s and H_w terms are irrelevant and can be set to zero. By using mean field approximation $\Delta = g_d\langle\Delta_{dj}\rangle$, we obtain the mean field Hamiltonian which breaks time-reversal symmetry T

$$H_{\text{MF}} = \sum_{j\sigma} -t(c_{j\sigma}^\dagger c_{j+1\sigma} + c_{j+1\sigma}^\dagger c_{j\sigma}) - \mu c_{j\sigma}^\dagger c_{j\sigma}$$

$$+\sum_j (\Delta^* c_{j\alpha}\sigma^x_{\alpha\beta} c_{j+1\beta} + \Delta c^\dagger_{j+1\beta}\sigma^x_{\beta\alpha} c^\dagger_{j\alpha}). \quad (12)$$

Along the quantization axis of σ_x, one easily find that the Hamiltonian is equivalent to two-component Kitaev model in one dimension with opposite Δ. Each sector is topological nontrivial when $|\mu| < 2|t|$, and can be transformed to each other through the transformation $c_\alpha \to \sigma^z_{\alpha\beta} c_\beta$. Four Majorana fermions are localized at the ends of two-leg ladder, resulting in four-fold degenerate ground state (see Fig. 3). The mean field result is a bit different from the bosonization one, where the ground state is two-fold degenerate. This discrepancy may come from the breaking of time-reversal symmetry or interactions of interchain.

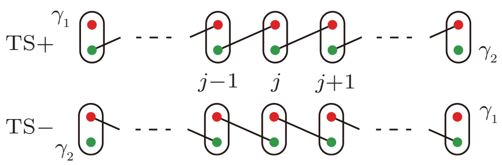

Fig. 3. Unpaired Majorana fermions in two Kitaev channels when $\mu = 0$ and $\Delta = \pm t$.

When the ground state is within the MS phase ($K_s > 1$, $K_c < 1$), the SDW order parameters decay the most slowly. For the special case $g_s = g_d = |w|/2$, the four fermion Hamiltonian can be written as

$$H_{\rm int} = -|w|\sum_j \Delta^\dagger_x(j)\Delta_x(j) \quad (13)$$

for negative w and

$$H_{\rm int} = -|w|\sum_j \Delta^\dagger_y(j)\Delta_y(j) \quad (14)$$

for positive w, where $\Delta_{x(y)}(j) = c^\dagger_{j\alpha}\sigma^{x(y)}_{\alpha\beta} c_{j+1\beta}$ is the SDW order parameter of $x(y)$ channel. The ground state is obviously characterized by nonzero SDW order Δ_x ($w<0$) or Δ_y ($w>0$), in accordance with the previous bosonization approach.

Specially, when the system is at half filling, chemical potential is zero $\mu = 0$ and $k_F = \pi/2$. Now umklapp scattering

$$R^\dagger_\uparrow R^\dagger_\downarrow L_\uparrow L_\downarrow + \text{H.c.} \sim \cos(\sqrt{8\pi}\phi_c) \quad (15)$$

takes place and contributes to a positive cosine term of ϕ_c. That will pin $\sqrt{2\pi}\phi_c = n\pi \pm \pi/2$ in the bulk, resulting in two-fold degeneracy in charge channel and a minimal kink $\sqrt{\pi/8}$ of ϕ_c at the edge. Such a kink of ϕ_c corresponds to a fractional charge at the edge

$$\rho = \int {\rm d}x(n_\uparrow + n_\downarrow) = \int {\rm d}x\sqrt{\frac{2}{\pi}}\partial_x\phi_c = \frac{1}{2}. \quad (16)$$

The ground state now becomes four-fold degenerate because of the degeneracy in both charge and spin channels. Meanwhile, the SDW orders oscillate as $(-)^x$, leading to the mean filed Hamiltonian which breaks $\mathbb{Z}_2$ and T symmetries simultaneously

$$H_{x(y)} = \sum_j c^\dagger_{j\alpha}\left[-t\sigma^0_{\alpha\beta} + (-)^j\Delta\sigma^{x(y)}_{\alpha\beta}\right] c_{j+1\beta} + \text{H.c.}, \quad (17)$$

where σ^0 is the identity matrix in spin basis and $\Delta \sim 2w/\pi$. Along the quantization axis of $\sigma^{x(y)}$, one easily finds that the Hamiltonian is equivalent to two-component SSH model. The ground state $|G_0\rangle$ is the direct product of both sectors, *i.e.*, one trivial and the other nontrivial, so the ground state is always topological nontrivial and two-fold degenerate with the phase transition point $\Delta = 0$. The degeneracy is also different from the bosonization approach, which may arise from the symmetry breaking. Specially, the critical point $|\Delta| = |t|$ corresponds to the fully dimerized SSH chain of intercell and intracell (see Fig. 4). Moreover, the $w>0$ and $w<0$ phases can be transformed to each other via spin rotation $(c_\uparrow, c_\downarrow)^{\rm T} \to {\rm e}^{{\rm i}\pi\sigma_z/4}(c_\uparrow, c_\downarrow)^{\rm T}$. In general, the mean field results of SDW orders, phase transition point, as well as the well-known fractional charge of soliton, are in good agreement with previous bosonization expectation of MS phase.

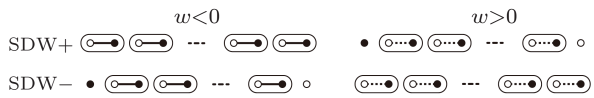

Fig. 4. Two-component fully dimerised SSH model at half filling in $\sigma^x_\pm$ ($w<0$) channel and $\sigma^y_\pm$ ($w>0$) channel. The ground state is the direct product of two sectors, always topological nontrivial with fractional charge at two ends.

6. High-order consideration

Until now, we take use of bosonization only to the first order. If we consider high-order perturbation, pair hopping will generate coupling of the form

$$\mathscr{H}_{\rm ss} \sim w\cos(\sqrt{8\pi}\phi_s)\cos(\sqrt{8\pi}\theta_s). \quad (18)$$

Under renormalization group operation, the couplings and Luttinger parameters in spin channel flow as

$$\begin{aligned}\frac{{\rm d}w}{{\rm d}l} &= 2(1-K_s-K_s^{-1})w,\\ \frac{{\rm d}g_\phi}{{\rm d}l} &= 2(1-K_s)g_\phi,\\ \frac{{\rm d}g_\theta}{{\rm d}l} &= 2(1-K_s^{-1})g_\theta,\\ \frac{{\rm d}K_s}{{\rm d}l} &= \frac{\pi^2}{2}(g_\theta^2 - K_s^2 g_\phi^2).\end{aligned} \quad (19)$$

The irrelevant w term apparently flows to zero since conjugated fields ϕ_s and θ_s cannot be localized simultaneously. The influence of second order $g^2_{\phi(\theta)}$ on K_s will become dramatic when $K_s \approx 1$, so the phase boundaries will be slightly modified depending on the relative magnitudes of g_ϕ and g_θ. In addition, situations at half filling are more subtle. Apart from the umklapp scattering, we have scattering

$$\mathscr{H}_{\rm m} = g_{\rm m}\cos(\sqrt{8\pi}\phi_c)\cos(\sqrt{8\pi}\theta_s), \quad (20)$$

where $g_{\mathrm{m}} = -w/\pi^2$ comes from the pair hopping, resulting in mixing of charge and spin channels. Comparing the sign of umklapp scattering strength and g_θ, we find $\mathscr{H}_{\mathrm{m}}$ is a win–win coupling that lowers energy in both channels. So it enhances the stability of ground states without spoiling the four-fold degeneracy.

In general, the high-order perturbation will not affect the degeneracy of ground state. The phase diagram from the first order bosonization is correct in this sense.

7. Conclusion

We have introduced a simple interacting fermionnic lattice model on the two-leg ladder system, which realizes topological nontrivial superconducting triplet pairing (TS) phase and Majorana-SDW (MS) phase. By using bosonization method, we obtain the full phase diagram of this model. The ground states of TS and MS phases (half filling) show fractional spin and fractional charge at the edges, respectively. In our number-conserving model, Majorana fermions can also be realized when pair hopping is favored, changing the sign of spin parity P_{s}. To understand the topological properties of the ground state and find its relation with the noninteracting case, we use mean field approximation to demonstrate that the TS phase can be interpreted as two Kitaev chains whereas the MS phase at half filling interpreted as two SSH chains. Despite its defects, mean field understanding of the phase diagram accords with the bosonization analysis to some extent, except for the ground state degeneracy. At last, our model is restricted to equal density with total charge as an even number. If the total particle number is odd, the spin parity P_{s} operator is non-Hermitian. We suggest the ground state degeneracy should be trivially doubled because of the reflection symmetry R. However, the non-equal density[50] and time-reversal symmetry breaking cases[51,52] remain an open question.

Acknowledgement

We thank Yan Chen for very fruitful discussions.

Appendix A: Luttinger parameters from bosonization

The Luttinger parameters are determined by the quadratic contributions to boson fields from interactions of superconducting pairing. Firstly, we write down the interaction forms of g_{s} and g_{d} channel

$$g_{\mathrm{d(s)}} \sim (c^\dagger_{j+1\downarrow} c^\dagger_{j\uparrow} \pm c^\dagger_{j+1\uparrow} c^\dagger_{j\downarrow})(c_{j\uparrow} c_{j+1\downarrow} \pm c_{j\downarrow} c_{j+1\uparrow}). \tag{A1}$$

One easily finds that the direct interaction terms give rise to quadratic contribution

$$n_{j\uparrow} n_{j+1\downarrow} + n_{j\downarrow} n_{j+1\uparrow} \sim \frac{1}{\pi}\left[(\partial\phi_{\mathrm{c}})^2 - (\partial\phi_{\mathrm{s}})^2\right], \tag{A2}$$

but the other four fermion terms $c^\dagger_{j+1\downarrow} c^\dagger_{j\uparrow} c_{j\downarrow} c_{j+1\uparrow} + \mathrm{H.c.}$ cannot be naively treated as $n_\uparrow n_\downarrow$. More precisely, by splitting fermions into left and right movers $c_\sigma(x) = R_\sigma \mathrm{e}^{\mathrm{i}k_{\mathrm{F}}x} + L_\sigma \mathrm{e}^{-\mathrm{i}k_{\mathrm{F}}x}$, we need to compute four terms from

$$-(R^\dagger_{j\uparrow} R_{j+1\uparrow} \mathrm{e}^{\mathrm{i}k_F} + L^\dagger_{j\uparrow} L_{j+1\uparrow} \mathrm{e}^{-\mathrm{i}k_F})(R^\dagger_{j+1\downarrow} R_{j\downarrow} \mathrm{e}^{-\mathrm{i}k_{\mathrm{F}}} + L^\dagger_{j+1\downarrow} L_{j\downarrow} \mathrm{e}^{\mathrm{i}k_{\mathrm{F}}}) \tag{A3}$$

and their conjugation. For the $R^\dagger_{j\uparrow} R_{j+1\uparrow} R^\dagger_{j+1\downarrow} R_{j\downarrow}$ term, it takes the bosonization form

$$-\frac{\mathrm{e}^{\mathrm{i}\sqrt{\pi}(\partial\phi_\uparrow - \partial\theta_\uparrow)} \mathrm{e}^{-\mathrm{i}\sqrt{\pi}(\partial\phi_\downarrow - \partial\theta_\downarrow)}}{(2\pi)^2} \sim -\left[1 + \mathrm{i}\sqrt{\pi}(\partial\phi_\uparrow - \partial\theta_\uparrow) - \frac{\pi}{2}(\partial\phi_\uparrow - \partial\theta_\uparrow)^2\right]\left[1 - \mathrm{i}\sqrt{\pi}(\partial\phi_\downarrow - \partial\theta_\downarrow) - \frac{\pi}{2}(\partial\phi_\downarrow - \partial\theta_\downarrow)^2\right], \tag{A4}$$

where we have expanded the exponential function to the second order. Neglecting the constant and fist order gradients (vanish in integral), we keep result to the second order

$$\frac{\pi}{2}(\partial\phi_\uparrow - \partial\theta_\uparrow - \partial\phi_\downarrow + \partial\theta_\downarrow)^2 = \pi(\partial\phi_{\mathrm{s}} - \partial\theta_{\mathrm{s}})^2, \tag{A5}$$

where we have used the definition of charge and spin boson $\phi_{\mathrm{c/s}} = (\phi_\uparrow \pm \phi_\downarrow)/\sqrt{2}$, $\theta_{\mathrm{c/s}} = (\theta_\uparrow \pm \theta_\downarrow)/\sqrt{2}$ and neglected the factor $1/(2\pi)^2$ at the moment. For the left movers, similarly we have

$$-L^\dagger_{j\uparrow} L_{j+1\uparrow} L^\dagger_{j+1\downarrow} L_{j\downarrow} \sim \pi(\partial\phi_{\mathrm{s}} + \partial\theta_{\mathrm{s}})^2. \tag{A6}$$

In contrast, the cross term $R^\dagger_\sigma R_\sigma L^\dagger_{\bar{\sigma}} L_{\bar{\sigma}} + \mathrm{H.c.}$ shows an phase factor $\mathrm{e}^{\pm \mathrm{i}2k_{\mathrm{F}}}$ and gives rise to

$$\frac{\cos 2k_{\mathrm{F}}}{\pi}\left[(\partial\phi_{\mathrm{c}})^2 + (\partial\theta_{\mathrm{s}})^2\right]. \tag{A7}$$

Finally, we show the explicit quadratic form of gradients in g_{s} and g_{d} channels

$$\begin{aligned} &\frac{2g_{\mathrm{s}}}{\pi}\left[\cos^2 k_{\mathrm{F}}(\partial\phi_{\mathrm{c}})^2 - (\partial\phi_{\mathrm{s}})^2 - \sin^2 k_{\mathrm{F}}(\partial\theta_{\mathrm{s}})^2\right] \\ &\frac{2g_{\mathrm{d}}}{\pi}\left[\sin^2 k_{\mathrm{F}}(\partial\phi_{\mathrm{c}})^2 + \sin^2 k_{\mathrm{F}}(\partial\theta_{\mathrm{s}})^2\right]. \end{aligned} \tag{A8}$$

Plugging with the bosonization form of H_0

$$\mathscr{H}_0 = \frac{v_{\mathrm{F}}}{2}\left[(\partial\phi_{\mathrm{c}})^2 + (\partial\theta_{\mathrm{c}})^2 + (\partial\phi_{\mathrm{s}})^2 + (\partial\theta_{\mathrm{s}})^2\right], \tag{A9}$$

we obtain the form of Luttinger parameters in Eq. (3).

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