Fluctuating specific heat in two-band superconductors*

Lei Qiao(乔雷)†, Cheng Chi(迟诚), and Jiangfan Wang(王江帆)

School of Physics, Peking University, Beijing 100871, China

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Theory of thermal fluctuations in two-band superconductors under an essentially homogeneous magnetic field is developed within the framework of the two-band Ginzburg–Landau theory. The fluctuating specific heat is calculated by using the optimized self-consistent perturbation approach and the results are applied to analyze the thermodynamic data of the iron-based superconductors $Ba_{1-x}K_xFe_2As_2$ with $x \sim 0.4$, which have been suggested to have a two-band structure by recent experiments. We estimate the fluctuation strength in this material and find that the specific heat is described well with the Ginzburg number $Gi = 4 \cdot 10^{-4}$. The influence of interband coupling strength is investigated and the result of the two-band Gaussian approximation approach is compared.

Keywords: fluctuating specific heat, two-band superconductors, vortex liquid, self-consistent approximation

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1. Introduction

Thermal fluctuations in superconductors have been studied for more than half a century.[1–3] Since the discovery of high T_c superconductors, superconducting fluctuations (SCF) have taken the center stage, particularly in the presence of a magnetic field. Due to thermal fluctuations, Cooper pairs may come into being while the systems are still in the normal state above T_c. These fluctuating Cooper pairs affect both thermodynamic and transport properties of superconductors.

Thermal fluctuations lead to smearing of the superconducting transition and broadening of the critical region. In the traditional cuprate superconductors, due to the high transition temperature, extreme anisotropy, and short coherence length, thermal fluctuations are strong and help to form various phases in the vortex state.[3–5] Physical properties with fluctuations in these superconductors have been extensively investigated both theoretically and experimentally, such as the electric and thermal conductivity,[6–9] Nernst effects,[10–12] magnetization,[13,14] and specific heat.[15,16]

To describe theoretically the fluctuations in superconductors, both microscopic theory and phenomenological Ginzburg–Landau (GL) theory are used.[1] GL theory is a good tool for describing the mesoscopic or macroscopic properties of superconductors near the transition temperature and is successful even in the case of superconductors with quite complicated band structures. In the GL theory, the fluctuating thermodynamic and transport properties of superconductors can be easily treated with the Gaussian approximation,[17,18] in which the coupling of fluctuating modes is neglected. However, the Gaussian approximation leads to a divergence at T_c and sometimes cannot well describe SCF especially in the superconductors with strong fluctuations.[19,20] To describe SCF better, one has to take the interaction term in GL free energy into account. A convenient treatment of the fluctuation interaction is the so-called Hartree approximation or Hartree–Fock (HF) approximation.[6,19,21,22]

Multi-band superconductors have been paid a great deal of attention since the discovery of superconductors MgB_2 and later iron-based superconductors.[23–25] The fluctuations in two-band superconductors have been studied above T_c by Gaussian approximation.[26,27] Recently, $Ba_{1-x}K_xFe_2As_2$ with $x \sim 0.4$, which is one of the typical iron-based superconductors, has been suggested to have a two-band structure by several experiments.[28–30] And recent experimental papers indicate that fluctuations in these superconductors play important roles in physical properties and are proposed to be related to the large iron isotope effect.[31–33] Therefore, it is natural to check whether or not the SCF in this kind of superconductor can be described theoretically by a two-band character.

Motivated by recent advances in the experiments of multiband superconductors, we examine the two-band fluctuating specific heat in the presence of magnetic fields. Based on the optimized self-consistent approximations,[34–36] a two-band GL theory considering fluctuations beyond Gaussian is developed. It is found that the experimental data of $Ba_{0.6}K_{0.4}Fe_2As_2$ can be fitted well by the two-band model and the fluctuation strength in this material is estimated. Besides, the relation between the transition temperature and the interband coupling is investigated. The results of the optimized self-consistent approximations and the Gaussian approximation are compared. Since our theory focuses on the vortex liquid, we do not consider the disorder effect.

The paper is organized as follows. In Section 2, the model and optimized self-consistent approximation are briefly intro-

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†Corresponding author. E-mail: leiqiao@pku.edu.cn

duced. In Section 3, details of the calculation are shown. In Section 4, we compare the result with experimental data and discuss the characters of the two-band model result. In Section 5, we summarize the study and give the conclusions.

2. Model and optimized self-consistent approximation

To describe the thermal fluctuations of the order parameter in two-band superconductors, we begin with the symmetric two-band GL free energy as a functional of the order parameter field ψ.[37] Generally, in the presence of magnetic field B, the free energy is

$$\begin{aligned} F = & \int_V \mathrm{d}^3 r \alpha \left\{ \sum_{n=1,2} \left[(\xi_{ab} \boldsymbol{\Pi} \psi_n)^\dagger (\xi_{ab} \boldsymbol{\Pi} \psi_n) \right.\right. \\ & + (\xi_c \partial_z \psi_n)^\dagger (\xi_c \partial_z \psi_n) + \varepsilon_0 |\psi_n|^2 + \left.\frac{\beta}{2\alpha} |\psi_n|^4 \right] \\ & \left. + \gamma (\psi_1^* \psi_2 + \psi_1 \psi_2^*) \right\}, \end{aligned} \tag{1}$$

where ψ_1 and ψ_2 are the two components of the superconducting order parameter. ξ_{ab} is the GL coherence length in the x–y plane and ξ_c is the z-direction coherence length. $\varepsilon_0 = T/T_0 - 1$ is the reduced temperature, here T_0 is the mean-field or bare transition temperature, which can be significantly larger than the measured critical temperature T_c due to strong thermal fluctuations on the mesoscopic scale and can be renormalized by considering more than the leading order contributions.[19,21,22] $\boldsymbol{\Pi} = -\mathrm{i}\boldsymbol{\nabla} + (2\pi/\phi_0)\boldsymbol{A}$ is the covariant momentum operator with the flux quantum ϕ_0, here $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential and we use the Landau gauge $\boldsymbol{A}(\boldsymbol{r}) = (0, Bx)$. γ is the strength of Josephson-like interband coupling. Note that if the interband coupling strength $\gamma = 0$, the two bands have a common bare critical temperature T_0. In the following sections, one will see that two distinct energy modes emerge due to the existence of the interband coupling.

In the case of strong type-II superconductors, when the external magnetic field applied along the z direction is much larger than the lower critical field $H_{c1}(T)$, the magnetization is smaller by a factor of κ^2 $(\kappa = \lambda/\xi \gg 1)$ than the external field. Therefore, it is a good approximation to take $B \approx \mu_0 H$.

To calculate the fluctuating specific heat, we use a method called the optimized self-consistent perturbation theory,[34,35] which is based on the "principle of minimal sensitivity".[36] The partition function is

$$Z = \int D(\psi^*, \psi) \exp\left(-\frac{F(\psi^*, \psi)}{T k_B}\right). \tag{2}$$

The GL free energy F can be divided into two parts: the optimized quadratic part $F_0(\psi^*, \psi, \mu)$ and a small perturbation $F_1(\psi^*, \psi, \mu)$, where μ is a variational parameter and is determined by minimization of the thermodynamic free energy density $f(\mu)$

$$\begin{aligned} f(\mu) &= -\frac{T k_B}{V} \ln Z \\ &= -\frac{T k_B}{V} \ln \int D(\psi^*, \psi) \exp\left(-\frac{F_0 + F_1}{T k_B}\right) \\ &\approx -\frac{T k_B}{V} \ln Z_0 + \frac{1}{V} \langle F_1 \rangle_0, \end{aligned} \tag{3}$$

where the zero-order partition function Z_0 and the first-order perturbation term $\langle F_1 \rangle_0$ are

$$Z_0 = \int D(\psi^*, \psi) \exp\left(-\frac{F_0}{T k_B}\right), \tag{4}$$

$$\langle F_1 \rangle_0 = Z_0^{-1} \int D(\psi^*, \psi) F_1 \exp\left(-\frac{F_0}{T k_B}\right). \tag{5}$$

By minimizing $f(\mu)$ with respect to μ, one can obtain the equation for μ

$$\frac{\mathrm{d} f(\mu)}{\mathrm{d}\mu} = 0, \tag{6}$$

which is called the self-consistent equation or gap equation.[19] Once the parameter μ is determined by Eq. (6), the functions $f(\mu)$, $F_0(\psi^*, \psi, \mu)$, and $F_1(\psi^*, \psi, \mu)$ are determined.

The method is very general. By introducing the parameter μ, we obtain a freedom to choose "the best" quadratic part.

3. Calculation of fluctuation specific heat

To calculate Z_0, we expand the order parameter via the Landau level (LL) eigenfunctions with the LL index N

$$\psi_n(\boldsymbol{r}) = \sum_{N,q,k} \varphi_{Nq}(x) \frac{\mathrm{e}^{\mathrm{i}qy}}{\sqrt{L_y}} \frac{\mathrm{e}^{\mathrm{i}kz}}{\sqrt{L_z}} a_{nNqk}, \tag{7}$$

where k is the wavevector in the z direction and q is the wavevector in the y direction. $\varphi_{Nq}(x)$ is the normalized eigenfunction of the N-th LL. a_{nNqk} represents the amplitude for each mode of the n-th band. Then the second-order term in Eq. (1) is written as

$$\begin{aligned} F_{2\mathrm{nd}} \equiv & \int_V \mathrm{d}^3 r \alpha \left\{ \sum_{n=1,2} [(\xi_{ab} \boldsymbol{\Pi} \psi_n)^\dagger (\xi_{ab} \boldsymbol{\Pi} \psi_n) \right. \\ & + (\xi_c \partial_z \psi_n)^\dagger (\xi_c \partial_z \psi_n) + \varepsilon_0 |\psi_n|^2] \\ & \left. + \gamma (\psi_1^* \psi_2 + \psi_1 \psi_2^*) \right\} \\ = & \sum_{N,q,k} \alpha \left\{ \left[\varepsilon_0 + 2h\left(N + \frac{1}{2}\right) + \xi_c^2 k^2 \right] \right. \\ & \times \left(|a_{1Nqk}|^2 + |a_{2Nqk}|^2 \right) \\ & \left. + \gamma \left(a_{1Nqk}^* a_{2Nqk} + a_{1Nqk} a_{2Nqk}^* \right) \right\}, \end{aligned} \tag{8}$$

where $h = 2\pi\xi_{ab}^2 \mu_0 H/\phi_0$ is a dimensionless magnetic field. In order to diagonalize Eq. (8), we expand the amplitudes a_{nNqk}

as

$$\begin{pmatrix} a_{1Nqk} \\ a_{2Nqk} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\operatorname{sgn}(\gamma) \end{pmatrix} b_{Nqk} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \operatorname{sgn}(\gamma) \end{pmatrix} \bar{b}_{Nqk}. \quad (9)$$

Thus, the diagonalized form of $F_{2\mathrm{nd}}$ is obtained as

$$F_{2\mathrm{nd}} = \sum_{Nqk} \alpha \left(E_{Nk} \left| b_{Nqk} \right|^2 + \bar{E}_{Nk} \left| \bar{b}_{Nqk} \right|^2 \right), \quad (10)$$

where the eigenvalues E_{Nk} and $\bar{E}_{Nk}$ are given by

$$E_{Nk} = \varepsilon + 2h \left(N + \frac{1}{2} \right) + \xi_c^2 k^2, \quad (11)$$

$$\bar{E}_{Nk} = \bar{\varepsilon} + 2h \left(N + \frac{1}{2} \right) + \xi_c^2 k^2. \quad (12)$$

Here, $\varepsilon = \varepsilon_0 - |\gamma|$ and $\bar{\varepsilon} = \varepsilon_0 + |\gamma|$ are the new reduced temperatures. One can see that due to the existence of the interband coupling, two distinct modes are produced: one mode possesses lower excitation energy E_{Nk} and has a lower reduced temperature ε, the other with higher energy mode $\bar{E}_{Nk}$ has a higher reduced temperature $\bar{\varepsilon}$. For convenience, we call the former as the low energy mode (LEM) and the latter as the high energy mode (HEM). Note that the thermodynamic free energy is divergent when $N \to \infty$. Usually, a momentum (or, equivalently, kinetic energy) cutoff, which corresponds to the lattice constant, is proposed to suppress the short wavelength fluctuating modes that cause the UV divergence.[38–40] Here, we follow Ref. [19] and apply the upper limit of the LL index N as $N_{\mathrm{cut}} = \frac{\Lambda}{2h} - 1$, $\bar{N}_{\mathrm{cut}} = \frac{\bar{\Lambda}}{2h} - 1$.

To investigate the specific heat of the superconductors, the coupling of the fluctuating modes in Eq. (1) is treated by using a variational method. Compared with the Gaussian approximation, this method enables us to consistently treat fluctuations in a wider temperature and field range in the vortex liquid region. Due to the existence of two bands, we divide the GL free energy into two parts by introducing two variational parameters μ and $\bar{\mu}$ for the LEM and the HEM, respectively: $F = F_0 + F_1$, where

$$\begin{aligned} F_0 = & \sum_{N,q,k} \alpha \left[\mu + 2hN + \xi_c^2 k^2 \right] \left| b_{Nqk} \right|^2 \\ & + \sum_{\bar{N},\bar{q},\bar{k}} \alpha \left[\bar{\mu} + 2h\bar{N} + \xi_c^2 \bar{k}^2 \right] \left| \bar{b}_{\bar{N}\bar{q}\bar{k}} \right|^2. \end{aligned} \quad (13)$$

Here, the parameter μ introduced is the renormalized mass term of the LEM, which corresponds to the energy gap of the spectrum when the magnetic field $h \to 0$, and the parameter $\bar{\mu}$ is for the case of the HEM. The free energy density is calculated as

$$\begin{aligned} f(\mu, \bar{\mu}) = & -\frac{T k_{\mathrm{B}}}{V} \ln \mathrm{Tr}_{b,\bar{b}} \left[\mathrm{e}^{-F/T k_{\mathrm{B}}} \right] \\ \approx & \; f_0 + \frac{\langle F_1 \rangle_0}{V}, \end{aligned} \quad (14)$$

where

$$\begin{aligned} \mathrm{Tr}_{b,\bar{b}}[\cdots] = & \int_{-\infty}^{+\infty} \prod_{N=0}^{N_{\mathrm{cut}}} \prod_{k,q} \mathrm{dRe} b_{Nqk} \, \mathrm{dIm} b_{Nqk} \\ & \times \int_{-\infty}^{+\infty} \prod_{\bar{N}=0}^{\bar{N}_{\mathrm{cut}}} \prod_{\bar{k},\bar{q}} \mathrm{dRe} \bar{b}_{\bar{N}\bar{q}\bar{k}} \, \mathrm{dIm} \bar{b}_{\bar{N}\bar{q}\bar{k}} [\cdots], \end{aligned} \quad (15)$$

and the zero-order free energy density f_0 and the first-order perturbation term $\langle F_1 \rangle_0$ are

$$\begin{aligned} f_0 = & -\frac{T k_{\mathrm{B}}}{V} \ln Z_0 \\ = & -\frac{T k_{\mathrm{B}}}{V} \ln \mathrm{Tr}_{b,\bar{b}} \left[\mathrm{e}^{-F_0/T k_{\mathrm{B}}} \right], \end{aligned} \quad (16)$$

$$\langle F_1 \rangle_0 = \frac{\mathrm{Tr}_{b,\bar{b}} \left[F_1 \, \mathrm{e}^{-F_0/T k_{\mathrm{B}}} \right]}{\mathrm{Tr}_{b,\bar{b}} \, \mathrm{e}^{-F_0/T k_{\mathrm{B}}}}. \quad (17)$$

Here the sum of possible q is $\mu_0 H L_x L_y / \phi_0$, which corresponds to the degree of degeneracy of each LL. Note that Z_0 takes the form of Gaussian functional integral. Thus, the zero-order free energy density f_0 can be obtained as

$$\begin{aligned} f_0 = & -\frac{T k_{\mathrm{B}} h}{(2\pi)^2 \xi_{ab}^2} \left\{ \int \mathrm{d}k \left[\frac{\Lambda}{2h} \ln \left(\frac{\pi T k_{\mathrm{B}}}{2h\alpha} \right) \right. \right. \\ & \left. - \ln \Gamma \left(\frac{\Lambda + \mu + \xi_c^2 k^2}{2h} \right) + \ln \Gamma \left(\frac{\mu + \xi_c^2 k^2}{2h} \right) \right] \\ & + \int \mathrm{d}\bar{k} \left[\frac{\bar{\Lambda}}{2h} \ln \left(\frac{\pi T k_{\mathrm{B}}}{2h\alpha} \right) - \ln \Gamma \left(\frac{\bar{\Lambda} + \bar{\mu} + \xi_c^2 \bar{k}^2}{2h} \right) \right. \\ & \left. \left. + \ln \Gamma \left(\frac{\bar{\mu} + \xi_c^2 \bar{k}^2}{2h} \right) \right] \right\}. \end{aligned} \quad (18)$$

In the derivation of Eq. (18), we use the equation $\prod_{N=0}^{N_{\mathrm{cut}}} (N + x) = \Gamma(N_{\mathrm{cut}} + 1 + x) / \Gamma(x)$, where $\Gamma(x)$ is the Gamma function. It should be pointed out that if we study the difference of specific heat under different magnetic fields, the contributions of the terms

$$\frac{T k_{\mathrm{B}} \Lambda}{2 (2\pi)^2 \xi_{ab}^2} \ln \left(\frac{\pi T k_{\mathrm{B}}}{2h\alpha} \right)$$

and

$$\frac{T k_{\mathrm{B}} \bar{\Lambda}}{2 (2\pi)^2 \xi_{ab}^2} \ln \left(\frac{\pi T k_{\mathrm{B}}}{2h\alpha} \right)$$

in Eq. (18) are zero. So, for simplicity, we neglect the two terms in the following calculations.

Substituting Eqs. (7) and (9) into the expression of $\langle F_1 \rangle_0$, one can calculate $\langle F_1 \rangle_0$ with the HF approximation[37]

$$\begin{aligned} \langle F_1 \rangle_0 = & -\alpha (\mu - h - \varepsilon) \sum_{N,q,k} \left\langle \left| b_{Nqk} \right|^2 \right\rangle_0 \\ & - \alpha (\bar{\mu} - h - \bar{\varepsilon}) \sum_{\bar{N},\bar{q},\bar{k}} \left\langle \left| \bar{b}_{\bar{N}\bar{q}\bar{k}} \right|^2 \right\rangle_0 \\ & + \frac{\beta}{2V} \left[\sum_{N,q,k} \left\langle \left| b_{Nqk} \right|^2 \right\rangle_0 + \sum_{\bar{N},\bar{q},\bar{k}} \left\langle \left| \bar{b}_{\bar{N}\bar{q}\bar{k}} \right|^2 \right\rangle_0 \right]^2. \end{aligned} \quad (19)$$

By using the relations

$$\sum_{N,q,k} \left\langle \left| b_{Nqk} \right|^2 \right\rangle_0 = \frac{V \partial f_0}{\alpha \partial \mu}$$

and

$$\sum_{\bar{N},\bar{q},\bar{k}} \left\langle \left|\bar{b}_{\bar{N}\bar{q}\bar{k}}\right|^2 \right\rangle_0 = \frac{V\partial f_0}{\alpha\partial\bar{\mu}},$$

$\langle F_1 \rangle_0$ is obtained as

$$\begin{aligned}\langle F_1 \rangle_0 = & -(\mu - h - \varepsilon)\frac{V\partial f_0}{\partial\mu} - (\bar{\mu} - h - \bar{\varepsilon})\frac{V\partial f_0}{\partial\bar{\mu}} \\ & + \frac{V\beta}{2\alpha^2}\left[\frac{\partial f_0}{\partial\mu} + \frac{\partial f_0}{\partial\bar{\mu}}\right]^2 . \end{aligned} \tag{20}$$

The two variational parameters μ and $\bar{\mu}$ are determined by minimizing free energy density $f(\mu,\bar{\mu})$ with respect to μ and $\bar{\mu}$, respectively, namely, $\partial f/\partial\mu = 0$ and $\partial f/\partial\bar{\mu} = 0$, which lead to the self-consistent equations

$$\mu = h + \varepsilon - \omega\frac{T}{T_0}\left[Y_\mu + \bar{Y}_{\bar{\mu}}\right], \tag{21}$$

$$\bar{\mu} = h + \bar{\varepsilon} - \omega\frac{T}{T_0}\left[Y_\mu + \bar{Y}_{\bar{\mu}}\right]. \tag{22}$$

The dimensionless fluctuating strength coefficient

$$\omega \equiv \frac{\beta}{\alpha^2}\frac{k_B T_0}{(2\pi)^2\xi_{ab}^2\xi_c} = \sqrt{2Gi_0}/\pi,$$

where Gi_0 is the bare Ginzburg number in the 3D case.[19] The parameter ω is a convenient measure of the fluctuating strength since the conventional Ginzburg number is small even for strongly fluctuating superconductors. Here, Y_μ and $\bar{Y}_{\bar{\mu}}$ are defined as

$$Y_\mu = \int_0^\infty \mathrm{d}\tilde{k}\left[\tilde{\psi}\left(\frac{\mu+\tilde{k}^2}{2h}\right) - \tilde{\psi}\left(\frac{\Lambda+\mu+\tilde{k}^2}{2h}\right)\right], \tag{23}$$

$$\bar{Y}_{\bar{\mu}} = \int_0^\infty \mathrm{d}\tilde{k}\left[\tilde{\psi}\left(\frac{\bar{\mu}+\tilde{k}^2}{2h}\right) - \tilde{\psi}\left(\frac{\bar{\Lambda}+\bar{\mu}+\tilde{k}^2}{2h}\right)\right], \tag{24}$$

where $\tilde{k} = \xi_c k$ is a dimensionless wavenumber. Here we introduce the digamma function $\tilde{\psi}(x) = \mathrm{d}\ln\Gamma(x)/\mathrm{d}x$. With Eqs. (21) and (22), one can easily obtain the relation $\bar{\mu} = \mu + 2|\gamma|$. As the temperature decreases, the gap μ of the LEM first reaches zero. The corresponding temperature is the renormalized critical temperature T_c, which is the real superconducting transition temperature. When the magnetic field $h \to 0$, we obtain the relation between the real critical temperature T_c and the bare transition temperature T_0

$$T_0 = \frac{T_c}{1 + |\gamma| + \omega_c\left[Y_0 + \bar{Y}_{0+2|\gamma|}\right]|_{h\to 0}}, \tag{25}$$

where

$$\omega_c = \frac{\beta}{\alpha^2}\frac{k_B T_c}{(2\pi)^2\xi_{ab}^2\xi_c} = \frac{\omega T_c}{T_0}$$

and the renormalized Ginzburg number is given by $Gi = (\omega_c\pi)^2/2$.

To study the relation between T_c and the interband coupling strength γ, we plot the $T_c(\gamma)$ curve using Eq. (25) with $T_0 = 35.2$ K in Fig. 1. Other parameters are $\omega = 0.009$, $\Lambda = 1.4$, and $\bar{\Lambda} = 1.2$. One can see that as the coupling strength γ increases, T_c almost increases linearly. A similar $T_c(\gamma)$ relation was also obtained in Ref. [41].

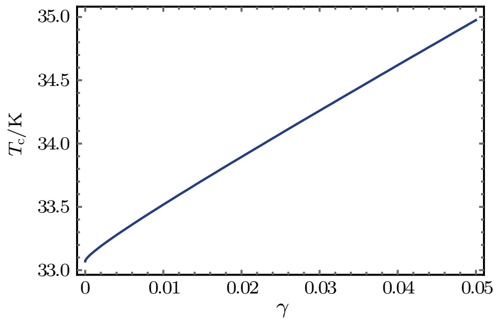

Fig. 1. (color online) The interband coupling strength dependence of T_c.

The specific heat caused by fluctuating Cooper pairs can be calculated by

$$\begin{aligned}C_s(T,\mu_0 H) &= -VT\frac{\partial}{\partial T}\frac{\partial f}{\partial T} \\ &= -VT\frac{\partial}{\partial T}\left[\frac{\partial f}{\partial T}|_{\mu\bar{\mu}} + \frac{\partial f}{\partial\mu}\frac{\partial\mu}{\partial T} + \frac{\partial f}{\partial\bar{\mu}}\frac{\partial\bar{\mu}}{\partial T}\right] \\ &= -VT\frac{\partial}{\partial T}\left[\frac{\partial f}{\partial T}|_{\mu\bar{\mu}}\right]. \end{aligned} \tag{26}$$

Here we use the fact $\partial_\mu f = \partial_{\bar{\mu}} f = 0\,(\partial_\mu = \partial/\partial\mu, \partial_{\bar{\mu}} = \partial/\partial\bar{\mu})$ in the last line. By combining Eqs. (21), (22), and (25), the specific heat with a renormalized critical temperature T_c is finally obtained as

$$\begin{aligned}& C_s(T,\mu_0 H) \\ = & \frac{VTk_B}{(2\pi)^2\xi_{ab}^2\xi_c}\Bigg(\left[Y_\mu + \bar{Y}_{\mu+2|\gamma|}\right] \\ & - \left[Y_0 + \bar{Y}_{0+2|\gamma|}\right]|_{h\to 0} - \frac{1+|\gamma|}{\omega_c}\Bigg) \\ & \times \frac{1+|\gamma| - \omega_c\left[Y_\mu + \bar{Y}_{\bar{\mu}} - \left[Y_0 + \bar{Y}_{0+2|\gamma|}\right]|_{h\to 0}\right]}{T_c + \omega_c T\left[\partial_\mu Y_\mu + \partial_{\bar{\mu}}\bar{Y}_{\bar{\mu}}\right]}. \end{aligned} \tag{27}$$

4. Results and discussion

The total specific heat including the contribution of fluctuating Cooper pairs and normal state part can be written as

$$C(T,\mu_0 H) = C_s(T,\mu_0 H) + C_n(T,\mu_0 H). \tag{28}$$

Here $C_n(T,\mu_0 H)$ stands for the field-independent background (phonon contribution, residual "linear term", etc).[42,43] In order to exclude the contribution of the field-independent background, we study the difference of specific heat $C(T,\mu_0 H)$ under different magnetic field $\mu_0 H$. In Fig. 2, the superconducting fluctuating specific heat $\Delta C(T,\mu_0 H)/T = [C(T,\mu_0 H) - C(T, 8\ \mathrm{T})]/T$ of $Ba_{0.6}K_{0.4}Fe_2As_2$ with various perpendicular magnetic fields is fitted. The points are the experimental data from Ref. [44]. The theoretical results (solid lines) are numerically calculated using Eq. (27). The fitting results give that GL coherence length $\xi_{ab} = 1.5$ nm, $\xi_c = 0.62$ nm, and other parameters are $\Lambda = 1.4$, $\bar{\Lambda} = 1.2$, and $\gamma = 0.045$. The fluctuating strength parameter $\omega_c = 0.009$ (corresponding to the Ginzburg number $Gi = 4 \times 10^{-4}$). The critical temperature is taken to be $T_c = 35.2$ K from the experimental data. It is observed that the theoretical results are in good agreement with the experimental data in a broad temperature range around the transition temperature.

One can see that the $\Delta C(T)/T$ curves with different magnetic fields in Fig. 2 show a typical broadening of superconducting transition, which reveals that fluctuation plays a significant role in the superconductors. This can also be reflected by the values of Ginzburg number Gi. It is much higher than that of the traditional BCS superconductors and has the same order of magnitude with the Ginzburg number in some cuprate superconductors.[1]

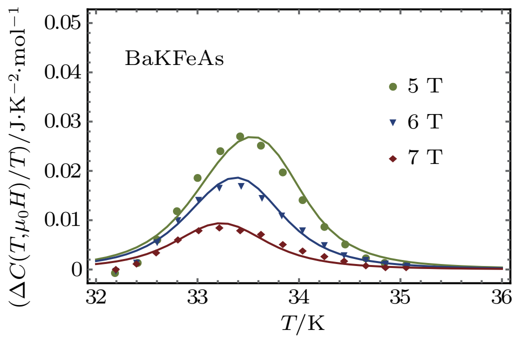

Fig. 2. (color online) The fitting result of the temperature dependence of the superconducting fluctuating specific heat $\Delta C(T,\mu_0 H)/T = [C(T,\mu_0 H) - C(T,8\ \mathrm{T})]/T$ with various perpendicular magnetic fields ($\mu_0 H = 5$ T, 6 T, 7 T). The points are the experiment data of $Ba_{1-x}K_xFe_2As_2$. The solid lines are the theoretical results.

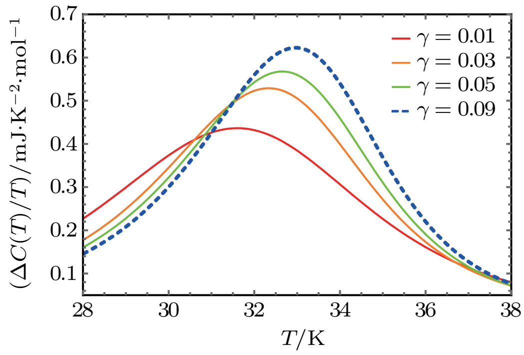

Fig. 3. (color online) The temperature dependence of the superconducting fluctuating specific heat $\Delta C(T)/T = [C(T,5\ \mathrm{T}) - C(T,8\ \mathrm{T})]/T$ with different interband coupling strength γ.

In order to investigate the dependence of the specific heat on the interband coupling strength γ, we plot $\Delta C(T)/T = [C(T,5\ \mathrm{T}) - C(T,8\ \mathrm{T})]/T$ as a function of the temperature with different coupling strength γ in Fig. 3. Other parameters are chosen to be the same as those used in Fig. 2. It is observed that as the coupling strength γ increases, the peaks of the $\Delta C(T)/T$ curves become higher. Besides, the superconducting transition temperatures of the $\Delta C(T)/T$ curves shift to higher temperatures when γ increases. This behavior is consistent with the result present in Fig. 1 that the interband coupling can enhance the transition temperature. In fact, as the coupling strength γ increases, the difference between the LEM and the HEM becomes larger. When the temperature decreases, the renormalized mass of the LEM can reach zero more easily. This is why the superconducting transition temperature is raised by the interband coupling.

Finally, we compare the specific heat with the result from the Gaussian approximation. Ignoring the quartic terms in Eq. (1) and following the above steps, one can obtain the result of two-band specific heat with Gaussian approximation as

$$\begin{aligned} & C_s(T,\mu_0 H) \\ = & \frac{2Tk_B}{(2\pi)^2 \xi_{ab}^2 \xi_c T_c} \\ & \times \left\{ (Y_{h+\varepsilon} + \bar{Y}_{h+\bar{\varepsilon}}) + \frac{T}{2} \left(\frac{\partial Y_{h+\varepsilon}}{\partial T} + \frac{\partial \bar{Y}_{h+\bar{\varepsilon}}}{\partial T} \right) \right\}. \end{aligned} \tag{29}$$

Here the function $Y_{h+\varepsilon}$ is defined as Eq. (23) with the variable μ being substituted with $h+\varepsilon$ and the function $\bar{Y}_{h+\bar{\varepsilon}}$ is defined in Eq. (24) with the variable $\bar{\mu}$ being substituted with $h+\bar{\varepsilon}$.

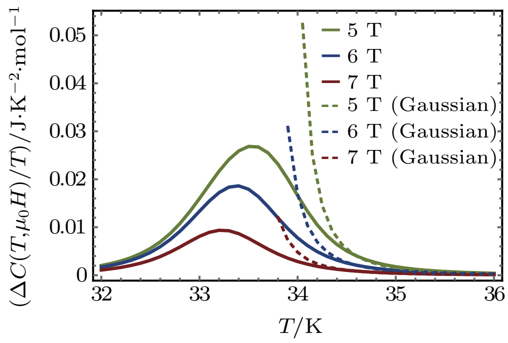

Fig. 4. (color online) The temperature dependence of the superconducting fluctuating specific heat $\Delta C(T,\mu_0 H)/T = [C(T,\mu_0 H) - C(T,8\ \mathrm{T})]/T$ with various perpendicular magnetic fields ($\mu_0 H = 5$ T, 6 T, 7 T). The solid lines are the results from the optimized self-consistent perturbation approach. The dashed lines are the results from the Gaussian approximation approach.

In Fig. 4, we show the results of the specific heat from the optimized self-consistent perturbation approach (solid lines) and the Gaussian approximation approach (dashed lines). Here T_c is taken to be 33.5 K, other parameters are the same as those used in Fig. 2. It is observed that the Gaussian approximation results can only present the behavior that $\Delta C(T)/T$ increases monotonically with decreasing temperature near the superconducting transition temperature T_c. Besides, the Gaussian result increases too fast as the temperature decreases around T_c. On the other hand, the result based on the self-consistent HF approximation is consistent with experimental data even when the temperature is far below T_c as shown in Fig. 2.

5. Summary and conclusions

We investigate the fluctuation-induced specific heat in two-band superconductors based on a symmetric two-band GL model with the optimized self-consistent perturbation approach. Due to the existence of interband coupling, two distinct energy modes emerge. Both the LEM with low excitation energy and the HEM with high excitation energy contribute to the fluctuating specific heat. Based on the HF approximation, we study the relation between the real critical temperature and the interband coupling and reach the conclusion that the interband coupling strength can enhance the superconducting transition temperature. By comparing with the recent experimental data, we demonstrate that the theoretical result can describe the experimental phenomena well and show that fluctuation plays an important role in the superconductors. Finally, we compare the specific heat with the result of Gaussian approximation. While the Gaussian theory can only give the result above T_c, the theory based on the optimized self-consistent approach can keep consistent with the experimental data even when the temperature is far below T_c.

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