Vol 11 No 1, January 2002
1009-1963/2002/11(01)/0005-04

Chinese Physics

Form invariance and Lie symmetry of equations of non-holonomic systems*

Wang Shu-Yong(王树勇) and Mei Feng-Xiang(梅凤翔)

Department of Applied Mechanics, Beijing Institute of Technology, Beijing 100081, *China*

(Received 2 July 2001; revised manuscript received 29 August 2001)

In this paper, we study the relation between the form invariance and Lie symmetry of non-holonomic systems. Firstly, we give the definitions and criteria of the form invariance and Lie symmetry in the systems. Next, their relation is deduced. We show that the structure equation and conserved quantity of the form invariance and Lie symmetry of non-holonomic systems have the same form. Finally, we give an example to illustrate the application of the result.

Keywords: analytical mechanics, non-holonomic mechanics, form invariance, Lie symmetry
PACC: 0320

1. Introduction

The conserved quantity of a mechanical system reflects its profound physical nature. In 1918, while carrying out research on the invariance of the Hamilton action under infinitesimal transformations of groups, Noether disclosed the relation between the conserved quantity and the symmetry of the mechanical system, and brought forward the famous Noether theory. Eventually, in 1980, Lutzky,[1] and Prince and Elizer[2,3] took examples to explain that the symmetry is not always the Noether form. This prompted people to refresh recognition of the symmetry, and to put forward a series of novel symmetry ideas, from which the Lie symmetry rapidly developed.[4−7] Research on the conserved quantity and symmetry has already become an important and active field of modern mathematical and physical science.

Form invariance is one of the equations of motion under infinitesimal transformations. It is different from the Noether symmetry and the Lie symmetry and can also lead to a conserved quantity under certain conditions. The form invariance of the Lagrange system,[8] the Appell equations,[9] the Nielsen equations[10] and the general holonomic systems[11] have been studied. In this paper, we will study the relation between the form invariance and Lie symmetry of non-holonomic systems.

2. Form invariance of equations

We let the position of a mechanical system be determined by n generalized coordinates $q_s (s=1,...,n)$ and its motion be subjected to g ideal non-holonomic constraints of the Chetaev type

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0 \quad (\beta=1,...,g). \tag{1}$$

Then its motion can be described by the differential equations as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+\sum_{\beta=1}^{g}\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \quad (s=1,...,n), \tag{2}$$

where $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is the Lagrangian, $Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ are the non-potential generalized forces, and λ_β are the multipliers. Introducing a Euler operator

$$E_s=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}-\frac{\partial}{\partial q_s} \quad (s=1,...,n), \tag{3}$$

then Eqs.(2) can be written in the form

$$E_s(L)=Q_s+\sum_{\beta=1}^{g}\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \quad (s=1,...,n). \tag{4}$$

Before integrating the equations of motion, we can find λ_β as functions of $t,\boldsymbol{q},\dot{\boldsymbol{q}}$

$$\lambda_\beta=\lambda_\beta\left(t,\boldsymbol{q},\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}\right) \quad (\beta=1,...,g). \tag{5}$$

Substituting this into Eqs.(4), we obtain

$$E_s(L)=Q_s+\Lambda_s \quad (s=1,...,n), \tag{6}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 19972010).

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{\beta=1}^{g} \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}. \quad (7)$$

Equations (6) are called the equations of motion of the holonomic system corresponding to the non-holonomic system expressed by Eqs.(1) and (2).

We introduce the infinitesimal transformations of time and coordinates as

$$t^* = t + \Delta t,$$
$$q_s^*(t^*) = q_s(t) + \Delta q_s \quad (s = 1, ..., n), \quad (8)$$

or their expansion formulae

$$t^* = t + \varepsilon \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (9)$$

where ε is an infinitesimal parameter and ξ_0, ξ_s are infinitesimal generators. Under transformations (9), L, Q_s, λ_β, f_β and Λ_s become $L^* = L\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)$, $Q_s^* = Q_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)$, $\lambda_\beta^* = \lambda_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)$, $f_\beta^* = f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)$, and $\Lambda_s^* = \Lambda_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)$.

Definition 1. If the form of constraint equations (1) and the differential equations (6) remain invariant under transformations (9), i.e.

$$f_\beta^* = f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = 0 \quad (\beta = 1, ..., g), \quad (10)$$

$$E_s(L^*) = Q_s^* + \Lambda_s^* \quad (s = 1, ..., n), \quad (11)$$

then the invariance is called the form invariance of the equations of motion for the non-holonomic system.

We introduce the differential operator of the infinitesimal generator

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \sum_{k=1}^{n} \xi_k \frac{\partial}{\partial q_k}, \quad (12)$$

and its extensions

$$X^{(1)} = X^{(0)} + \sum_{k=1}^{n} (\dot{\xi}_k - \dot{q}_k \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_k}, \quad (13)$$

$$X^{(2)} = X^{(1)} + \sum_{k=1}^{n} ((\dot{\xi}_k - \dot{q}_k \dot{\xi}_0)^{\cdot} - \ddot{q}_k \dot{\xi}_0) \frac{\partial}{\partial \ddot{q}_k}. \quad (14)$$

Expanding L^*, Q_s^*, Λ_s^* and f_β^*, we have

$$L^* = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon[X^{(1)}(L)] + O(\varepsilon^2), \quad (15)$$

$$Q_s^* = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon[X^{(1)}(Q_s)] + O(\varepsilon^2), \quad (16)$$

$$\Lambda_s^* = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon[X^{(1)}(\Lambda_s)] + O(\varepsilon^2), \quad (17)$$

$$f_\beta^* = f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon[X^{(1)}(f_\beta)] + O(\varepsilon^2). \quad (18)$$

From Eqs.(10)–(18), we obtain the following.

Criterion 1. For a non-holonomic system expressed by Eqs.(1) and (2), if the infinitesimal generators ξ_0 and ξ_s satisfy the following relations

$$X^{(1)}(f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})) = 0 \quad (\beta = 1, ..., g), \quad (19)$$

$$E_s(X^{(1)}(L)) - X^{(1)}(Q_s) - X^{(1)}(\Lambda_s) = 0 \quad (s = 1, ..., n), \quad (20)$$

then it is form invariant under the infinitesimal transformations (9).

Proof: Substituting Eqs.(18) into Eqs.(10), using Eqs.(1), substituting Eqs.(15)–(17) into Eqs.(11), using Eqs.(6), and ignoring ε^2 and the higher infinitesimal terms, Eqs.(19) and (20) will be obtained.

3. Lie symmetry of equations

The basic idea of the Lie theory is to keep the equations of motion (6) invariant under infinitesimal transformations (9). Equations (6) are rewritten as

$$F_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = E_s(L) - Q_s - \Lambda_s = 0 \quad (s = 1, ..., n). \quad (21)$$

Definition 2. If the constraint equations (1) and the differential equations (21) remain invariant under infinitesimal transformations (9), i.e. Eqs.(10) hold and

$$F_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, \frac{\mathrm{d}^2\boldsymbol{q}^*}{\mathrm{d}t^{*2}}\right) = 0 \quad (s = 1, ..., n), \quad (22)$$

then the invariance is called the Lie symmetry of the equations of motion for the non-holonomic system.

Expanding F_s, we have

$$F_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, \frac{\mathrm{d}^2\boldsymbol{q}^*}{\mathrm{d}t^{*2}}\right)$$
$$= F_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) + \varepsilon X^{(2)}(F_s) + O(\varepsilon^2)$$
$$(s = 1, ..., n). \quad (23)$$

Criterion 2. For the non-holonomic systems (1) and (2), if the infinitesimal generators ξ_0 and ξ_s satisfy Eqs.(19) and the following relations

$$X^{(2)}(E_s(L)) - X^{(1)}(Q_s) - X^{(1)}(\Lambda_s) = 0$$
$$(s = 1, ..., n), \quad (24)$$

then the invariance is the Lie symmetry of the non-holonomic system.

Proof: Substituting Eqs.(21) and (23) into Eqs.(22), we can obtain Eqs.(24).

4. Relation between form invariance and Lie symmetry

From the deduction of Eqs.(11) and (22), it can be seen that the form invariance is generally different from the Lie symmetry. According to the criteria 1 and 2, the relation between the form invariance and the Lie symmetry is actually that between $E_s(X^{(1)}(L))$ and $X^{(2)}(E_s(L))$. We have

$$
\begin{aligned}
&E_s(X^{(1)}(L))\\
=&E_s\left(\xi_0\frac{\partial L}{\partial t}+\sum_{k=1}^{n}\xi_k\frac{\partial L}{\partial q_k}+\sum_{k=1}^{n}(\dot\xi_k-\dot q_k\dot\xi_0)\frac{\partial L}{\partial \dot q_k}\right)\\
=&E_s(\xi_0)\frac{\partial L}{\partial t}+\xi_0E_s\left(\frac{\partial L}{\partial t}\right)+\dot\xi_0\frac{\partial^2 L}{\partial t\partial\dot q}+\frac{\partial \xi_0}{\partial \dot q_s}\left(\frac{\partial L}{\partial t}\right)^{\cdot}\\
&+\sum_{k=1}^{n}\left[\xi_kE_s\left(\frac{\partial L}{\partial q_k}\right)+E_s(\xi_k)\frac{\partial L}{\partial q_k}\right.\\
&\left.+\dot\xi_k\frac{\partial^2 L}{\partial q_k\partial\dot q_s}+\frac{\partial \xi_k}{\partial \dot q_s}\left(\frac{\partial L}{\partial q_k}\right)^{\cdot}\right]\\
&+\sum_{k=1}^{n}\left[E_s(\dot\xi_k-\dot q_k\dot\xi_0)\frac{\partial L}{\partial \dot q_k}+(\dot\xi_k-\dot q_k\dot\xi_0)E_s\left(\frac{\partial L}{\partial \dot q_k}\right)\right.\\
&\left.+(\dot\xi_k-\dot q_k\dot\xi_0)^{\cdot}\frac{\partial^2 L}{\partial \dot q_k\partial\dot q_s}+\frac{\partial}{\partial \dot q_s}(\dot\xi_k-\dot q_k\dot\xi_0)\left(\frac{\partial L}{\partial \dot q_k}\right)^{\cdot}\right]\\
=&X^{(2)}(E_s(L))+Y^{(1)}(L)+\frac{\partial \xi_0}{\partial \dot q_s}\left(\frac{\partial L}{\partial t}\right)^{\cdot}\\
&+\sum_{k=1}^{n}\frac{\partial \xi_k}{\partial \dot q_s}\left(\frac{\partial L}{\partial q_k}\right)^{\cdot}+\sum_{k=1}^{n}\left(\frac{\partial \dot\xi_k}{\partial \dot q_s}-\dot q_k\frac{\partial \dot\xi_0}{\partial \dot q_s}\right)\left(\frac{\partial L}{\partial \dot q_k}\right)^{\cdot}\\
&(s=1,...,n),
\end{aligned}\tag{25}
$$

where

$$
\begin{aligned}
Y^{(1)}(L)=&E_s(\xi_0)\frac{\partial L}{\partial t}+\sum_{k=1}^{n}E_s(\xi_k)\frac{\partial L}{\partial q_k}\\
&+\sum_{k=1}^{n}E_s(\dot\xi_k-\dot q_k\dot\xi_0)\frac{\partial L}{\partial \dot q_k}.
\end{aligned}\tag{26}
$$

The relation between the form invariance and the Lie symmetry is given by Eqs.(25), and then we have the following.

Proposition 1. If the equations of motion are form invariant under infinitesimal transformations (9), and the following relations hold

$$
\begin{aligned}
&Y^{(1)}(L)+\frac{\partial \xi_0}{\partial \dot q_s}\left(\frac{\partial L}{\partial t}\right)^{\cdot}\\
&+\sum_{k=1}^{n}\frac{\partial \xi_k}{\partial \dot q_s}\left(\frac{\partial L}{\partial q_k}\right)^{\cdot}+\sum_{k=1}^{n}\left(\frac{\partial \dot\xi_k}{\partial \dot q_s}-\dot q_k\frac{\partial \dot\xi_0}{\partial \dot q_s}\right)\left(\frac{\partial L}{\partial \dot q_k}\right)^{\cdot}=0\\
&(s=1,...,n),
\end{aligned}\tag{27}
$$

then the equations are also Lie symmetrical.

If the infinitesimal generators ξ_0 and ξ_s do not explicitly contain $\dot{\boldsymbol{q}}$, i.e.

$$
\begin{aligned}
&t^*=t+\varepsilon\xi_0(t,\boldsymbol{q}),\\
&q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q})\quad(s=1,...,n),
\end{aligned}\tag{28}
$$

then Eqs.(25) become

$$
\begin{aligned}
&E_s(X^{(1)}(L))\\
=&X^{(2)}(E_s(L))-\frac{\partial \xi_0}{\partial q_s}\frac{\partial L}{\partial t}-\sum_{k=1}^{n}\frac{\partial \xi_k}{\partial q_s}\frac{\partial L}{\partial q_k}\\
&-\sum_{k=1}^{n}\left(\ddot q_k\frac{\partial \xi_0}{\partial q_s}+\frac{\partial q_k}{\partial q_s}\ddot\xi_0\right)\frac{\partial L}{\partial \dot q_k}\\
&+\sum_{k=1}^{n}\left(\frac{\partial \xi_k}{\partial q_s}-\dot q_k\frac{\partial \xi_0}{\partial q_s}\right)\left(E_k(L)+\frac{\partial L}{\partial q_k}\right)\\
=&X^{(2)}(E_s(L))-\frac{\partial \xi_0}{\partial q_s}\dot L-\ddot\xi_0\frac{\partial L}{\partial \dot q_s}\\
&+\sum_{k=1}^{n}\left(\frac{\partial \xi_k}{\partial q_s}-\dot q_k\frac{\partial \xi_0}{\partial q_s}\right)(Q_k+\Lambda_k)\\
&(s=1,...,n).
\end{aligned}\tag{29}
$$

Corollary 1. If the equations of motion are form invariant under infinitesimal transformations (28), and the following relations hold

$$
\begin{aligned}
&\frac{\partial \xi_0}{\partial q_s}\dot L+\ddot\xi_0\frac{\partial L}{\partial \dot q_s}\\
=&\sum_{k=1}^{n}\left(\frac{\partial \xi_k}{\partial q_s}-\dot q_k\frac{\partial \xi_0}{\partial q_s}\right)(Q_k+\Lambda_k)\\
&(s=1,...,n),
\end{aligned}\tag{30}
$$

then the equations are also Lie symmetrical.

Corollary 2. If the equations of motion are form invariant under the infinitesimal transformations

$$
\begin{aligned}
&t^*=t+\varepsilon(at+b),\\
&q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q})\quad(s=1,...,n),
\end{aligned}\tag{31}
$$

and the following relations hold

$$
\sum_{k=1}^{n}\frac{\partial \xi_k}{\partial q_s}(Q_k+\Lambda_k)=0\quad(s=1,...,n),\tag{32}
$$

then the equations are also Lie symmetrical.

5. Conserved quantity of form invariance and Lie symmetry

For the non-holonomic systems, the form invariance and the Lie symmetry do not always lead to a conserved quantity. The conditions in which these can lead to a conserved quantity are given by following propositions.

Proposition 2. If the non-holonomic system given by Eqs.(1) and (2) is form invariant under infinitesimal transformations (9), and there is a gauge function $G_{\rm F} = G_{\rm F}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the structure equation

$$L\dot{\xi}_0 + X^{(1)}(L) + \sum_{k=1}^{n}(\xi_k - \dot{q}_k\xi_0)(Q_k + \Lambda_k) + \dot{G}_{\rm F} = 0, \tag{33}$$

then it will have the following conserved quantity

$$I = L\xi_0 + \sum_{k=1}^{n}(\xi_k - \dot{q}_k\xi_0)\frac{\partial L}{\partial \dot{q}_k} + G_{\rm F} = \text{const}. \tag{34}$$

Proposition 3. If the non-holonomic system given by Eqs.(1) and (2) is Lie symmetrical under infinitesimal transformations (9), and there is a gauge function $G_{\rm L} = G_{\rm L}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the structure equation

$$L\dot{\xi}_0 + X^{(1)}(L) + \sum_{k=1}^{n}(\xi_k - \dot{q}_k\xi_0)(Q_k + \Lambda_k) + \dot{G}_{\rm L} = 0, \tag{35}$$

then it will have the following conserved quantity

$$I = L\xi_0 + \sum_{k=1}^{n}(\xi_k - \dot{q}_k\xi_0)\frac{\partial L}{\partial \dot{q}_k} + G_{\rm L} = \text{const}. \tag{36}$$

From propositions 2 and 3, we know that the structure equation and the conserved quantity of the form invariance and Lie symmetry of non-holonomic systems have the same form.

6. Illustrated example

Let us study the form invariance and Lie symmetry of the Appell–Hamel example. The Lagrangian of the system is

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3,$$

and the non-holonomic constraint is

$$f = \dot{q}_1^2 + \dot{q}_2^2 - \dot{q}_3^2 = 0.$$

The non-potential generalized forces are

$$Q_1 = Q_2 = Q_3 = 0.$$

Equations (2) give

$$m\ddot{q}_1 = 2\lambda\dot{q}_1, \quad m\ddot{q}_2 = 2\lambda\dot{q}_2, \quad m\ddot{q}_3 + mg = -2\lambda\dot{q}_3.$$

We obtain

$$\lambda = -\frac{mg}{4\dot{q}_3},$$

$$\Lambda_1 = -\frac{1}{2}mg\frac{\dot{q}_1}{\dot{q}_3}, \quad \Lambda_2 = -\frac{1}{2}mg\frac{\dot{q}_2}{\dot{q}_3}, \quad \Lambda_3 = \frac{1}{2}mg.$$

Taking

$$\xi_0 = 1, \quad \xi_1 = \xi_2 = \xi_3 = 0,$$

formulae (19) and (20) can be seen to hold up, which corresponds to a form invariance of the system. From corollary 2, we know it is also Lie symmetrical. Taking $G_{\rm N} = G_{\rm L} = 0$, the structure equations (33) and (35) hold, so the system has a conserved quantity

$$I = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + mgq_3 = \text{const}.$$

References

[1] Lutzky M 1979 *J. Phys. A: Math. Gen.* **12** 973

[2] Prince G E and Eliezer C J 1980 *J. Phys.* A **13** 815

[3] Prince G E and Eliezer C J 1981 *J. Phys.* A **14** 578

[4] Mei FX 2000 *ASME Appl. Mech. Rev.* **53** 283

[5] Zhang R C, Chen X W and Mei F X 2000 *Chin. Phys.* **9** 561

[6] Zhang R C, Chen X W and Mei F X 2000 *Chin. Phys.* **9** 801

[7] Zhang R C, Chen X W and Mei F X 2001 *Chin. Phys.* **10** 12

[8] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120

[9] Mei F X 2001 *Chin. Phys.* **10** 177

[10] Wang S Y and Mei F X 2001 *Chin. Phys.* **10** 373

[11] Mei F X 2001 *J. Beijing Inst.Technol.* **21** 535 (in Chinese)