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ON THE FORM INVARIANCE OF NIELSEN EQUATIONS*

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The definition and criterion of the form invariance of Nielsen equations are given. The relation between the form invariance and the Noether symmetry is studied. Some examples are given to illustrate the application of the result.

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I. INTRODUCTION

The symmetry of a mechanical system has a close relation with the conserved quantity. There are two main symmetry methods to seek the conserved quantity of the mechanical system: Noether symmetry and Lie symmetry. Noether symmetry is an invariance of the Hamilton action under the infinitesimal transformations of groups. Lie symmetry is an invariance of the differential equations under the infinitesimal transformations of groups.[1]

In this paper, we study the invariance of Nielsen equations; this invariance is called the form invariance. It means that the form of Nielsen equations remains invariant under the infinitesimal transformations of groups. But it is different from the Noether symmetry. The form invariance can lead to a conserved quantity only under certain conditions.

II. FORM INVARIANCE OF NIELSEN EQUATIONS

Let the position of a mechanical system be determined by the n generalized coordinates $q_s(s = 1,...,n)$, and its motion be described by the Nielsen equations as follows:

$$\frac{\partial \dot{L}}{\partial \dot{q}_s} - 2\frac{\partial L}{\partial q_s} = 0 \qquad (s = 1,...,n), \tag{1}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian. For the holonomic conservative system, the holonomic system in which the generalized forces have generalized potential, the system of inverse problem of the Lagrange mechanics,[2,3] the nonholonomic system whose corresponding holonomic system with Eqs.(1), the Chaplygin system with Helmholtz potential[4,5] and the nonholonomic potential system in which the free motion is realized,[6,7] their equations of motion can be written in the form of Eq.(1).

Introduce the infinitesimal transformations of time and coordinates as

$$\begin{aligned} &t^* = t + \Delta t, \\ &q_s^*(t^*) = q_s(t) + \Delta q_s \quad (s = 1,...,n) \end{aligned} \tag{2}$$

or their expansion formulae

$$\begin{aligned} &t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}), \\ &q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}), \end{aligned} \tag{3}$$

where ε is an infinitesimal parameter and ξ_0, ξ_s are infinitesimal generators. Under the transformations (3), the function $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ becomes $L^* = L\left(t^*, \boldsymbol{q}^*, \dfrac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)$.

Definition If the form of Nielsen equations (1) remains invariant under the infinitesimal transformations (3), i.e.,

$$\frac{\partial \dot{L}^*}{\partial \dot{q}_s} - 2\frac{\partial L^*}{\partial q_s} = 0 \quad (s = 1,...,n), \tag{4}$$

then the invariance is called the form invariance of Nielsen equations.

Expanding L^*, we have

$$\begin{aligned} L^* =& L\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\left[\frac{\partial L}{\partial t}\xi_0\right. \\ &\left. + \sum_{s=1}^{n}\frac{\partial L}{\partial q_s}\xi_s + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\right] + O(\varepsilon^2). \end{aligned} \tag{5}$$

Criterion If there are a constant κ and a function $G_{\mathrm{L}} = G_{\mathrm{L}}(t, \boldsymbol{q})$, that the infinitesimal generators

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ξ_0, ξ_s satisfy the following relation

$$\frac{\partial L}{\partial t}\xi_0 + \sum_{s=1}^{n}\frac{\partial L}{\partial q_s}\xi_s + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)$$
$$= kL - \dot{G}_{\mathrm{L}}, \tag{6}$$

then the Nielsen equations (1) are form invariant under the infinitesimal transformations (3).

Proof Taking the Nielsen operator

$$N_s = \frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t} - 2\frac{\partial}{\partial q_s},$$

from Eqs.(5) and (6), we have

$$N_s(L^*) = N_s(L) + kN_s(L) - \varepsilon N_s(\dot{G}_{\mathrm{L}}) = 0.$$

III. FORM INVARIANCE AND NOETHER SYMMETRY

The Noether theory points out[8−10] that for a system determined by Lagrangian L, if the infinitesimal generators satisfy the Noether identity

$$\frac{\partial L}{\partial t}\xi_0 + \sum_{s=1}^{n}\frac{\partial L}{\partial q_s}\xi_s + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) + L\dot{\xi}_0 = -\dot{G}_{\mathrm{N}}, \tag{7}$$

where $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{q})$, then the symmetry is called Noether symmetry. The Noether symmetry leads to the following conserved quantity

$$I = L\xi_0 + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{N}} = \text{const.} \tag{8}$$

The form invariance of Nielsen equations can be either a Noether symmetry or not. We have the following proposition.

Proposition If Nielsen equations (1) are form invariant under the infinitesimal transformations (3), and a gauge function $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{q})$ exists, satisfying

$$kL - \dot{G}_{\mathrm{L}} + L\dot{\xi}_0 = -\dot{G}_{\mathrm{N}}, \tag{9}$$

then the form invariance is a Noether symmetry; or else it is not.

Proof Substituting Eq.(6) into Eq.(7), we can obtain Eq.(9).

IV. ILLUSTRATIVE EXAMPLES

Example 1 We produce the generalized energy integral and circle integral by the form invariance.

If

$$\frac{\partial L}{\partial t} = 0, \tag{10}$$

in Eq.(6), let $\xi_0 = 1, \xi_s = 0$ and $k = 0, G_{\mathrm{L}} = 0$, and by Eq.(7), we have $G_{\mathrm{N}} = 0$, and by Eq.(8) we obtain the integral

$$I = L - \sum_{s=1}^{N}\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s = \text{const.} \tag{11}$$

This is a generalized energy integral.

If

$$\frac{\partial L}{\partial q_1} = 0, \tag{12}$$

let $\xi_0 = 0, \xi_1 = 1, \xi_s = 0 (s = 2, ..., n)$ and $k = 0, G_{\mathrm{L}} = 0$, then we have Eq.(6), by Eq.(9), we have $G_{\mathrm{N}} = 0$, and by Eq.(8) we obtain the integral

$$I = \frac{\partial L}{\partial \dot{q}_1} = \text{const.} \tag{13}$$

This is a circle integral.

Example 2 The Lagrangian of the one-degree-freedom system is

$$L = t^2\left(\frac{1}{2}\dot{q}^2 - \frac{1}{6}q^6\right). \tag{14}$$

Let us study the form invariance and the Noether symmetry of Nielsen equations.

We take the infinitesimal generators

$$\xi_0 = t, \quad \xi = -\frac{1}{2}q. \tag{15}$$

Substituting these into Eq.(6), we obtain

$$2t^2\left(\frac{1}{2}\dot{q}^2 - \frac{1}{6}q^6\right) - t^2q^5\left(-\frac{1}{2}q\right)$$
$$+ t^2\dot{q}\left(-\frac{1}{2}\dot{q} - \dot{q}\right) = kL - \dot{G}_{\mathrm{L}},$$

i.e.,

$$-L = kL - \dot{G}_{\mathrm{L}}.$$

Taking

$$k = -1, \quad G_{\mathrm{L}} = 0, \tag{16}$$

we see that formula (6) holds. Therefore, infinitesimal generators (15) correspond to a form invariance of Nielsen equations. Substituting Eqs.(15) and (16) into Eq.(9), we obtain

$$G_{\mathrm{N}} = 0. \tag{17}$$

So, the form invariance leads to a Noether symmetry. In this case, integral (8) gives

$$I = -\frac{1}{2}t^3\dot{q}^2 - \frac{1}{2}t^2q\dot{q} - \frac{1}{6}t^3q^6 = \text{const.} \tag{18}$$

Example 3 The Lagrangian of the plane Kepler problem is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \mu(q_1^2 + q_2^2)^{-\frac{1}{2}}. \tag{19}$$

Let us study its form invariance and Noether symmetry.

We take the infinitesimal generators

$$\xi_0 = t, \quad \xi_1 = \frac{2}{3}q_1, \quad \xi_2 = \frac{2}{3}q_2. \tag{20}$$

Substituting these into Eq.(6), we obtain

$$-\frac{2}{3}q_1^2\mu(q_1^2+q_2^2)^{-\frac{3}{2}} - \frac{2}{3}q_2^2\mu(q_1^2+q_2^2)^{-\frac{3}{2}} + \dot{q}_1\left(\frac{2}{3}\dot{q}_1 - \dot{q}_1\right) + \dot{q}_2\left(\frac{2}{3}\dot{q}_2 - \dot{q}_2\right) = kL - \dot{G}_{\rm L},$$

i.e.,

$$-\frac{2}{3}L = kL - \dot{G}_{\rm L}.$$

Taking

$$k = -\frac{2}{3}, \quad G_{\rm L} = 0, \tag{21}$$

then formula (6) holds; therefore, it is form invariant.

Substituting Eqs.(20) and (21) into Eq.(9), we obtain

$$\dot{G}_{\rm N} = -\frac{1}{3}L. \tag{22}$$

From this, we cannot seek a function $G_{\rm N}$. So, the form invariance is not a Noether symmetry.

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