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FORM INVARIANCE OF APPELL EQUATIONS*

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The form invariance of Appell equations of holonomic mechanical systems under the infinitesimal transformations of groups is studied. The definition and the criterion of the form invariance of Appell equations are given. This form invariance can lead to a conserved quantity under certain conditions.

Keywords: Appell equations, form invariance, Noether symmetry, conserved quantity
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I. INTRODUCTION

Appell equations are very important in analytical mechanics.[1−5] However, to this day, the solution of Appell equations is still not clear. In this paper, studying a form invariance of Appell equations under the infinitesimal transformations of groups and comparing the invariance with Noether symmetry, we will seek the integral of the equations. First, we establish the Appell equations and study their form invariance under the infinitesimal transformations. Next, we transform the Appell equations into Lagrange equations. Finally, we compare the form invariance with Noether symmetry to seek the conserved quantity of the Appell equations.

II. APPELL EQUATIONS

Let the position of a mechanical system be determined by the n generalized coordinates $q_s(s=1,\cdots,n)$, its energy of acceleration be

$$S=S(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}), \tag{1}$$

and the generalized forces be

$$Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}). \tag{2}$$

The Appell equations can be written in the form

$$\frac{\partial S}{\partial \ddot{q}_s}=Q_s, \quad (s=1,\cdots,n). \tag{3}$$

Introducing a function A as

$$A=S-\sum_{s=1}^{n}\ddot{q}_sQ_s, \tag{4}$$

Equations (3) become[4]

$$\frac{\partial A}{\partial \ddot{q}_s}=0, \quad (s=1,\cdots,n). \tag{5}$$

III. FORM INVARIANCE OF APPELL EQUATIONS

Introducing the infinitesimal transformations of time and coordinates as

$$\begin{aligned}&t^*=t+\Delta t,\\ &q_s^*(t^*)=q_s(t)+\Delta q_s \quad (s=1,\cdots,n),\end{aligned} \tag{6}$$

or their expansion formulae

$$\begin{aligned}&t^*=t+\varepsilon\xi_0(t,\boldsymbol{q}),\\ &q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q}),\end{aligned} \tag{7}$$

where ε is an infinitesimal parameter and ξ_0,ξ_s are infinitesimal generators, we then have

$$\begin{aligned}\frac{\mathrm{d}q_s^*}{\mathrm{d}t^*}&=\frac{\mathrm{d}q_s+\varepsilon\mathrm{d}\xi_s}{\mathrm{d}t+\varepsilon\mathrm{d}\xi_0}\\ &=\dot{q}_s+\varepsilon(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)+O(\varepsilon^2),\\ \frac{\mathrm{d}^2q_s^*}{\mathrm{d}t^{*2}}&=\ddot{q}_s+\varepsilon[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)^\bullet-\ddot{q}_s\dot{\xi}_0]+O(\varepsilon^2).\end{aligned} \tag{8}$$

Definition If the form of Appell equations (5) keeps an invariance under the infinitesimal transformations (7), then the invariance is called the form invariance of Appell equations.

By definition, the form invariance can be written in the form

$$\frac{\partial A^*}{\partial \ddot{q}_s}=\frac{\partial A}{\partial \ddot{q}_s}+O(\varepsilon^2)=O(\varepsilon^2) \quad (s=1,\cdots,n), \tag{9}$$

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where

$$A^* = A\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, \frac{\mathrm{d}^2\boldsymbol{q}^*}{\mathrm{d}t^{*2}}\right). \tag{10}$$

We have the following criterion.

Criterion For a given function A, if the infinitesimal generators ξ_0, ξ_s satisfy the following formula

$$\frac{\partial}{\partial \ddot{q}_k}\left\{\frac{\partial A}{\partial t}\xi_0 + \sum_{s=1}^{n}\frac{\partial A}{\partial q_s}\xi_s + \sum_{s=1}^{n}\frac{\partial A}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\right\} = 0$$
$$(k = 1, \cdots, n), \tag{11}$$

then Appell equations are form invariant.

Proof We have

$$A^* = A(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) + \varepsilon\left\{\frac{\partial A}{\partial t}\xi_0 + \sum_{s=1}^{n}\frac{\partial A}{\partial q_s}\xi_s + \sum_{s=1}^{n}\frac{\partial A}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) + \sum_{s=1}^{n}\frac{\partial A}{\partial \ddot{q}_s}[(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)^{\bullet} - \ddot{q}_s\dot{\xi}_0]\right\} + O(\varepsilon^2),$$

$$\frac{\partial A^*}{\partial \ddot{q}_k} = \varepsilon\frac{\partial}{\partial \ddot{q}_k}\left\{\frac{\partial A}{\partial t}\xi_0 + \sum_{s=1}^{n}\frac{\partial A}{\partial q_s}\xi_s + \sum_{s=1}^{n}\frac{\partial A}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\right\} + O(\varepsilon^2) = O(\varepsilon^2).$$

IV. LAGRANGE REALIZATION OF APPELL EQUATIONS

In order to compare the form invariance of Appell equations with Noether symmetry, it is necessary that one transforms Appell equations into Lagrange equations.

For a given function A, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \frac{\partial A}{\partial \ddot{q}_s} \equiv \sum_{k=1}^{n} A_{sk}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\ddot{q}_k + B_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \tag{12}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian. A necessary and sufficient condition for system (12) to be self-adjoint is that all conditions (13) satisfied[6]

$$A_{sk} = A_{ks},$$
$$\frac{\partial A_{sk}}{\partial \dot{q}_r} = \frac{\partial A_{rk}}{\partial \dot{q}_s},$$
$$\frac{\partial B_s}{\partial \dot{q}_k} + \frac{\partial B_k}{\partial \dot{q}_s} = 2\left(\frac{\partial}{\partial t} + \sum_{r=1}^{n}\dot{q}_r\frac{\partial}{\partial q_r}\right)A_{sk},$$
$$\frac{\partial B_s}{\partial q_k} - \frac{\partial B_k}{\partial q_s} = \frac{1}{2}\left(\frac{\partial}{\partial t} + \sum_{r=1}^{n}\dot{q}_r\frac{\partial}{\partial q_r}\right)\left(\frac{\partial B_s}{\partial \dot{q}_k} - \frac{\partial B_k}{\partial \dot{q}_s}\right),$$
$$(s, k, r = 1, \cdots, n). \tag{13}$$

Under these conditions, Appell equations can be expressed by Lagrange equations. The most general structure of the Lagrangian L is

$$L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = K(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \sum_{k=1}^{n} D_k(t, \boldsymbol{q})\dot{q}_k + C(t, \boldsymbol{q}), \tag{14}$$

where the n+2 functions K, D_k ,C are a solution of the linear, generally overdetermined system of partial differential equations

$$\frac{\partial^2 K}{\partial \dot{q}_s \partial \dot{q}_k} = A_{ks}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{15}$$

$$\frac{\partial D_s}{\partial q_k} - \frac{\partial D_k}{\partial q_s} = \frac{1}{2}\left(\frac{\partial B_s}{\partial \dot{q}_k} - \frac{\partial B_k}{\partial \dot{q}_s}\right) + \frac{\partial^2 K}{\partial q_s \partial \dot{q}_k} - \frac{\partial^2 K}{\partial \dot{q}_s \partial q_k} \equiv Z_{sk}(t, \boldsymbol{q}), \tag{16}$$

$$\frac{\partial C}{\partial q_s} = \frac{\partial D_s}{\partial t} - B_s - \frac{\partial K}{\partial q_s} + \frac{\partial^2 K}{\partial \dot{q}_s \partial t} + \sum_{k=1}^{n}\left[\frac{\partial^2 K}{\partial q_s \partial \dot{q}_k} + \frac{1}{2}\left(\frac{\partial B_s}{\partial \dot{q}_k} - \frac{\partial B_k}{\partial \dot{q}_s}\right)\right]\dot{q}_k \equiv W_s(t, \boldsymbol{q}), \tag{17}$$

and the solution is[6]

$$K(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{s=1}^{n}\sum_{k=1}^{n}\dot{q}_s\int_0^1 \mathrm{d}\tau' \cdot \left\{\left[\int_0^1 \mathrm{d}\tau A_{sk}(t, \boldsymbol{q}, \tau\dot{\boldsymbol{q}})\right]\dot{q}_k\right\}(t, \boldsymbol{q}, \tau'\dot{\boldsymbol{q}}), \tag{18}$$

$$D_s = \sum_{k=1}^{n}\left\{\int_0^1 \mathrm{d}\tau\tau Z_{sk}(t, \tau\boldsymbol{q})\right\}q_k, \tag{19}$$

$$C = \sum_{k=1}^{n}\left\{\int_0^1 \mathrm{d}\tau W_k(t, \tau\boldsymbol{q})\right\}q_k. \tag{20}$$

For given S and Q_s, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = \frac{\partial S}{\partial \ddot{q}_s} \equiv \sum_{k=1}^{n} a_{sk}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\ddot{q}_k + b_s(t, q, \dot{q}) = 0, \tag{21}$$

where T is the kinetic energy of the system. In a similar manner, we can seek the function T.

V. FORM INVARIANCE AND NOETHER SYMMETRY

The Noether theory points out that for a system determined by the Lagrangian L and the generalized forces Q_s, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s \qquad (s = 1, \cdots, n), \tag{22}$$

if the infinitesimal generators ξ_0, ξ_s and the gauge function $G = G(t, \boldsymbol{q})$ satisfy the Noether identity

$$L\dot{\xi}_0 + \frac{\partial L}{\partial t}\xi_0 + \sum_{s=1}^{n}\frac{\partial L}{\partial q_s}\xi_s + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) + \sum_{s=1}^{n} Q_s(\xi_s - \dot{q}_s\xi_0) = -\dot{G}, \tag{23}$$

then the system possesses the following conserved quantity[7−14]

$$I = L\xi_0 + \sum_{s=1}^{n}\frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G = \text{const.} \tag{24}$$

Proposition For the infinitesimal generators ξ_0, ξ_s, under which Appell equations are form invariant, if there exists a gauge function $G = G(t, \boldsymbol{q})$ satisfying identity (23), then the form invariance will lead up to the conserved quantity (24).

VI. ILLUSTRATIVE EXAMPLE

A system has

$$A = \frac{1}{2}\ddot{q}^2 + \ddot{q}\dot{q}. \tag{25}$$

Let us study its form invariance and the conserved quantity.

First, we study the form invariance of Appell equations. Formula (11) gives

$$\frac{\partial}{\partial \ddot{q}}\{\ddot{q}(\dot{\xi} - \dot{q}\dot{\xi}_0)\} = 0,$$

i.e.

$$\dot{\xi} - \dot{q}\dot{\xi}_0 = 0. \tag{26}$$

Equation (26) has the following solutions

$$\xi_0 = 0, \quad \xi = 1, \tag{27}$$

$$\xi_0 = 1, \quad \xi = 0, \tag{28}$$

$$\xi_0 = t, \quad \xi = q, \tag{29}$$

Next, we transform Appell equations into Lagrange equations. By virtue of Eq.(25), we have

$$S = \frac{1}{2}\ddot{q}^2, \quad Q = -\dot{q}. \tag{30}$$

Formula (21) gives

$$T = L = \frac{1}{2}\dot{q}^2. \tag{31}$$

Finally, we investigate whether or not the form invariances (27)–(29) are Noether symmetry. Substituting (31) into Eq.(23), we have

$$\frac{1}{2}\dot{q}^2\xi_0 + \dot{q}(\dot{\xi} - \dot{q}\dot{\xi}_0) - \dot{q}(\xi - \dot{q}\xi_0) = -\dot{G}. \tag{32}$$

Substituting Eqs.(27)–(29) into Eq.(32), we obtain respectively

$$-\dot{q} = -\dot{G}, \tag{33}$$

$$\frac{3}{2}\dot{q}^2 = -\dot{G}, \tag{34}$$

$$\frac{3}{2}\dot{q}^2 t - q\dot{q} = -\dot{G}. \tag{35}$$

By virtue of Eq.(33), we obtain

$$G = q. \tag{36}$$

By Eq.(34) or Eq.(35), we cannot seek the function G. Therefore, the generators (27) correspond to a Noether symmetry. By Eq.(24), we obtain a conserved quantity

$$I = \dot{q} + q = \text{const.}$$

In addition, the equation of motion of the system can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{\partial A}{\partial \ddot{q}} = \ddot{q} + \dot{q} = 0. \tag{37}$$

But, system (37) is not self-adjoint. Equation (37) is rewritten as

$$e^t(\ddot{q} + \dot{q}) = 0. \tag{38}$$

Then it is self-adjoint. In this case, the function A becomes

$$A = e^t\left(\frac{1}{2}\ddot{q}^2 + \ddot{q}\dot{q}\right).$$

Formula (11) gives

$$e^t(\ddot{q} + \dot{q})\xi_0 + e^t(\dot{\xi} - \dot{q}\dot{\xi}_0) = 0,$$

i.e.

$$\dot{\xi} - \dot{q}\dot{\xi}_0 = 0,$$

it is identical with Eq.(26). In this case, by Eqs.(14)–(20), we can seek the Lagrangian as

$$L = \frac{1}{2}e^t\dot{q}^2. \tag{39}$$

Substituting Eq.(39) into Eq.(23), we obtain

$$\frac{1}{2}t^2\dot{q}^2\dot{\xi}_0 + t\dot{q}^2\xi_0 + t^2\dot{q}(\dot{\xi} - \dot{q}\dot{\xi}_0) = -\dot{G}. \tag{40}$$

Substituting Eq.(27) into Eq.(40), we obtain

$$G = 0.$$

Therefore the form invariance (27) of Eqs.(37) is a Noether symmetry, and the corresponding conserved quantity is

$$I = e^t\dot{q} = \text{const.} \tag{41}$$

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