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**SPECIAL TOPIC — Non-Hermitian physics**

# Quantum simulation of $\tau$-anti-pseudo-Hermitian two-level systems

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Different from the Hermitian case, non-Hermitian (NH) systems have novel properties and strongly relate to open and dissipative quantum systems. In this work, we investigate how to simulate $\tau$-anti-pseudo-Hermitian systems in a Hermitian quantum device using linear combinations of unitaries and duality quantum algorithm. Specifying the $\tau$ to time-reversal ($T$) and parity-time-reversal ($PT$) operators, we construct the two NH two-level systems, design quantum circuits including three qubits, and decide the quantum gates explicitly in detail. We also calculate the success probabilities of the simulation. Experimental implementation can be expected in small quantum simulator.

**Keywords:** quantum simulation, linear combination of unitaries, non-Hermitian, anti-pseudo-Hermitian

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## 1. Introduction

Non-Hermitian (NH) systems, closely relating to open and dissipative quantum systems,[1–9] are more common in real physical world than closed Hermitian systems, and are attracting increased research interest in recent decades. The featured properties of NH systems enlighten various investigations both in fundamental researches and in applications.[10–33] parity-time-reversal ($PT$) symmetric systems and the anti-symmetric counterparts can be investigated experimentally in different physical systems by different methods, such as NMR,[34] optical micro-cavities,[35] photons–magnons coupled optical systems,[36] quantum photonics,[37] metal-semiconductor complex systems,[38] other optical systems,[39] etc.

Besides the $PT$ symmetric systems,[11–17] pseudo-Hermitian (PH) systems,[10,18–22,40–46] and their anti-symmetric counterparts,[31,47–58] a class of NH systems of $\tau$-anti-pseudo-Hermitian ($\tau$-APH) Hamiltonian was defined and investigated in 2002.[21] Both the $\tau$-APH and its anti-linear operator $\tau$ relevant to the symmetry are important to thoroughly investigate the sufficient and necessary conditions of an NH Hamiltonian having real spectrums.[21]

Obeying Feynman's thinking that nature should be simulated by itself,[59] quantum simulation has become an efficient way to investigate various phenomena and properties by controllable quantum systems. Besides Hermitian systems,[60–71] NH systems can be investigated effectively and efficiently in the way of quantum simulation, such as $PT$-symmetric and $P$-pseudo-Hermitian systems,[34,37,72–76] their anti-symmetric counterparts,[40,57,58] and other novel systems.[77–79]

In this work, we investigate quantum simulation of $\tau$-anti-pseudo-Hermitian two-level systems, proposing in detail both for a general case and for special cases that the $\tau$ being $T$ and $PT$. We use linear combination of unitaries (LCU) in the scheme of duality quantum computing[80] and the unitary-expansion (UE) technique[6,7] to achieve the quantum simulation in an indeterministic way, optimize the quantum circuit, and calculate the success probability. At last, we discuss how to generalize our method to simulate a general $\tau$-APH high-level system, and suggest experimental systems for implementation in the near future.

## 2. The $\tau$-anti-pseudo-Hermitian symmetry

A $\tau$-APH Hamiltonian $H_\tau$, defined by Mostafazadeh,[21] satisfies

$$\tau^{-1}{H_\tau}^\dagger\tau = H_\tau, \tag{1}$$

where $\tau : \mathcal{H} \to \mathcal{H}$ is a fixed antilinear anti-Hermitian invertible transformation (or automorphism) on the Hilbert space $\mathcal{H}$. $\tau : \mathcal{H} \to \mathcal{H}$ is said to be an antilinear operator if

$$\tau(a|\xi\rangle + b|\zeta\rangle) = a^*\tau|\xi\rangle + b^*\tau|\zeta\rangle, \tag{2}$$

where $*$ is the complex conjugation, $\forall\, a, b \in \mathbb{C}$ and $\forall\, \xi, \zeta \in \mathcal{H}$.

Further, $\tau : \mathcal{H} \to \mathcal{H}$ is anti-Hermitian if

$$\langle\zeta|\tau|\xi\rangle = \langle\xi|\tau|\zeta\rangle, \tag{3}$$

$\forall\, \xi$ and $\zeta \in \mathcal{H}$. $\tau$ can be the time-reversal ($T$) operator, parity-time-reversal ($PT$) operator, etc. Notice that the anti-symmetry of the PH Hamiltonian introduced in Ref. [40] is relevant to a linear Hermitian automorphism (such as the parity operator $P$), so it is completely different from the $\tau$-anti-PH Hamiltonian that relevant to an antilinear anti-Hermitian automorphism $\tau$[21] we investigate in this work.

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## 3. The $\tau$-APH two-level systems

Since $P$, $T$ and their combinations are basic symmetries attracting increased research interest,[81–85] we illustrate the NH systems by specifying $\tau$ as operators $T$ and $PT$ in two-dimensional cases. Notice that $PT$-anti-pseudo-Hermiticity and $PT$-symmetry are two distinguishable symmetries, of which Hamiltonians can be classified by the relevant mathematical relations of symmetries, i.e., $PTH(PT)^{-1} = H$ and $PTH^{\dagger}(PT)^{-1} = H$, respectively.

### 3.1. $\tau = T$

In the two-dimensional case, the general form of $T$-APH Hamiltonian $H_T$ can be decided by setting $\tau = T$ and the symmetry in Eq. (1)

$$H_T = \begin{pmatrix} u_1 + \mathrm{i}v_1 & s + \mathrm{i}w \\ s + \mathrm{i}w & u_2 + \mathrm{i}v_2 \end{pmatrix}, \tag{4}$$

where $s$, $w$, $u_k$ and $v_k$ $(k = 1, 2)$ are real parameters, and the time-reversal operator $T$ has an effect as the complex conjugate operating on its right operators and state-vectors.

### 3.2. $\tau = PT$

In the two-dimensional case, the general form of $PT$-APH Hamiltonian $H_T$ can be decided by setting $\tau = PT$ and the symmetry in Eq. (1)

$$H_{PT} = \begin{pmatrix} u + \mathrm{i}v & s_1 + \mathrm{i}w_1 \\ s_2 + \mathrm{i}w_2 & u + \mathrm{i}v \end{pmatrix}, \tag{5}$$

where $u$ and $v$, $s_k$ and $w_k$ $(k = 1, 2)$ are real parameters, and the parity operator $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

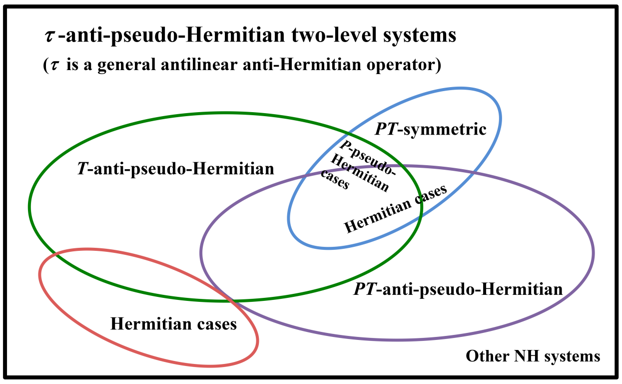


**Fig. 1.** Subsets of the $\tau$-anti-pseudo-Hermitian systems. The intersection of the $T$-anti-pseudo-Hermitian and the $PT$-symmetric systems are $P$-pseudo Hermitian, while that of the $PT$-anti-pseudo-Hermitian and $PT$-symmetric systems are Hermitian.

We show the relations between typical NH two-level sets in Fig. 1: $\tau$-APH ($\tau = T$ or $PT$), $PT$-symmetric, $P$-pseudo-Hermitian systems, and Hermitian cases. In special cases, the $\tau$-APH systems become $P$-pseudo-Hermitian or trivial Hermitian systems.

## 4. Quantum simulation of $T$- and $PT$-anti-pseudo-Hermitian two-level systems

It is the Hermiticity that guarantees the unitarity of the time-evolution of a Hermitian system, which can be simulated in the Hilbert space of the same dimensions. However, the two-dimensional time-evolutionary operator,

$$E(t) = \mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_\tau} \tag{6}$$

that we will simulate, is not unitary. Thus, one qubit cannot achieve the simulation. A $\tau$-APH subsystem will be constructed in a larger Hilbert space and simulated by qubits using LCU and duality quantum algorithm.

The exceptional points (EPs),[86,87] consist of the points in the parametric space, lead the eigenvalues of $H_\tau$ to be zero. Except the EPs, our quantum simulation is applicable to the whole parametric space (including the neighborhoods of EPs). Because the NH systems are simulated in a Hermitian system, the conventional Hilbert–Schmidt inner product will be used.

### 4.1. LCU and duality quantum algorithm

We now propose the quantum simulation of the time-evolution in Eq. (6) based on LCU and duality quantum algorithm[80] in a Hermitian quantum device.

LCU and duality quantum algorithm were proposed in 2002,[80] and they were developed fast,[88–93] becoming one of the strongest tools in designing quantum algorithms.[94] Besides to accelerate and optimize Hermitian quantum simulation, we developed LCU to simulate NH systems,[34,37,40,57,58,72–76] other novel systems,[77,78] and time-dependent non-unitary operators.[6] Our unitary expansion (UE) technique is used to realize the LCU in the detailed cases.

### 4.2. UE of the time-evolutionary operator

In two-dimensional cases,[6] the maximum of the number of UE-terms is four in general, and it can be reduced to two in some special cases when the parameters meet the phase matching conditions. We now applied the UE techniques to the non-unitary time-evolutionary operator. In detail,

$$\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_\tau} = f_0\sigma_0 + f_1\sigma_1 + f_2(\mathrm{i}\sigma_2) + f_3\sigma_3, \tag{7}$$

where the Pauli matrices $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $f_k = |f_k|\,\mathrm{e}^{\mathrm{i}\theta_k}$ $(k = 0, 1, 2, 3)$ are the UE-parameters, which are time-dependent complex-functions of the parameters in $H_\tau$. Notice that $|f_k|$'s are not limited by the normalization. When $\tau$ is specified to $T$ or $PT$, Eq. (7) has three UE-terms only

$$\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_T} = g_0\sigma_0 + g_1\sigma_1 + g_2\sigma_3, \tag{8}$$

$$\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_{PT}} = h_0\sigma_0 + h_1\sigma_1 + h_2(\mathrm{i}\sigma_2), \tag{9}$$

where the explicit forms of $g_k$ and $h_k$ ($k=0,1,2$) for $\tau=T$ or $PT$ are shown in Appendix A (i) and (ii), respectively.

We set $g_\mathrm{s}$ and $h_\mathrm{s}$ the summation of $|g_k|$'s and $|h_k|$'s ($k=0,\ldots,3$) for convenience respectively as

$$g_\mathrm{s} = \sum_{k=0}^{2}\sqrt{|g_k|^2} \quad \text{and} \quad h_\mathrm{s} = \sum_{k=0}^{2}\sqrt{|h_k|^2}. \tag{10}$$

Now, we are able to simulate the non-unitary evolution in Eq. (6) using LCU. A qudit or a qudit-qubit-hybrid device is able to achieve the simulation, and qudits take advantages over qubits in some quantum algorithm,[95] e.g., it reaches a higher accuracy to solve the eigenvalue problem using quantum phase estimation algorithm by qudits[96] than by qubits.[97] However, we focus on quantum simulations using qubits here, since qubit-quantum computers are available technologies.

### 4.3. Qubits simulation of *T*-APH two-level systems

We first take a $T$-anti-pseudo-Hermitian system as an example for illustration. One evolutionary qubit $e$ and two ancillary qubits $a$ are able to simulate the time-evolution of a general $T$-APH system by our theory. The evolutionary qubit will evolve as Eq. (8) with the assistance of the ancillary qubits in a probabilistic way. We will use a six-dimensional Hilbert subspace extended by $|lj\rangle_a|k\rangle_e$'s ($lj=00,01,10$ and $k=0,1$) to achieve the simulation, and the rest two dimensions are spared. Notice that the success probability using six dimensions is larger than that using the full eight dimensions,[6] while the later one can also achieve the simulation.

The quantum circuit to simulate a $T$-APH two-level system is shown in Fig. 2. At the beginning, the whole system is initialized to a pure state $|00\rangle_a|0\rangle_e$, and the evolutionary qubit $e$ will be prepared in an arbitrary state $|\psi\rangle_e$ as need by a single-qubit rotation $R_\psi$. The two ancillary qubits will assist the evolutionary qubit to evolve governed by the $T$-APH Hamiltonian. In the first block, we will spare the two dimensions relevant to $|11\rangle_a|k\rangle_e$ ($k=0,1$) to prepare the six-dimensional subspace, and assign the three UE-parameters $g_k$'s ($k=0,1,2$) to $|lj\rangle_a|k\rangle_e$'s ($lj=00,01,10$ and $k=0,1$). During this process, the first and second ancillary qubits are swapped, and then two single-qubit operators $G_1$ and $-\sigma_3$ rotate them. A controlled-NOT gate follows, of which the first and the second qubits are the target and control qubits, respectively. After operated by $G_2$, the first qubit is swapped again with the other ancillary qubit. The explicit forms of the two single-qubit operators are

$$G_1 = \frac{1}{g_\mathrm{s}}\begin{pmatrix} \dfrac{g_1}{\sqrt{|g_\mathrm{s}|^2-|g_1|^2}} & -\sqrt{|g_\mathrm{s}|^2-|g_1|^2} \\ & g_1^* \end{pmatrix}, \tag{11}$$

$$G_2 = \frac{1}{\sqrt{|g_\mathrm{s}|^2-|g_1|^2}}\begin{pmatrix} g_2^* & g_0 \\ -g_0^* & g_2 \end{pmatrix}, \tag{12}$$

where $g_0$, $g_1$, $g_2$ and $g_\mathrm{s}$ are those in Eqs. (8) and (10).

In the second block of Fig. 2, the three UE-terms in Eq. (8) will be generated. Notice that the unit-matrix $C_{00-\sigma_0}$ has theoretical meaning to show our method clearly, while no operation is needed in practice. The other two jointly-controlled gates are necessary, and their explicit forms are

$$C_{01-\sigma_1} = \begin{pmatrix} \sigma_0 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_0 & 0 \\ 0 & 0 & 0 & \sigma_0 \end{pmatrix}, \tag{13}$$

$$C_{10-G_3} = \begin{pmatrix} \sigma_0 & 0 & 0 & 0 \\ 0 & \sigma_0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_0 \end{pmatrix}. \tag{14}$$

Now the three UE terms are generated, and we will combine them together in the next step.

The third block aims at combining the three UE-terms. Five operators will be applied in series, i.e., a first swapping gate, a Hadamard gate $H_2$, a second swapping gate, a single-qubit rotation $G_3$, and a third swapping gate, as shown in Fig. 2, where

$$G_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}. \tag{15}$$

Now, the whole system evolves to a superposition

$$\frac{1}{\sqrt{3}g_\mathrm{s}}\left(|00\rangle_a\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_T}|\psi\rangle_e + g_\mathrm{s}\sum_{k=01,10}|k\rangle_a|s_k\rangle_e\right), \tag{16}$$

$|s_k\rangle_e$'s will be discarded after quantum measurements if $|k\rangle_a$ ($k=01,10$) is output, so their explicit forms are not given. Notice that the three UE-terms are combined to the time-evolution operator and superposed in the first term relevant to $|00\rangle_a$.

Finally, quantum measurements are performed on the ancillary qubits. If it outputs the state $|00\rangle_a$, the evolutionary qubit will evolve as the time-evolution $\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_T}|\psi\rangle_e$ governed by the $T$-APH Hamiltonian in Eq. (4), with a success probability of

$$\frac{1}{3g_\mathrm{s}^2}{}_e\langle\psi|\mathrm{e}^{\mathrm{i}\frac{t}{\hbar}H_T^\dagger}\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_T}|\psi\rangle_e. \tag{17}$$

If one of the rest two results of $|01\rangle_a$ or $|01\rangle_a$ is measured, the simulation will be terminated. The whole process will then be started over until $|00\rangle_a$ is obtained. Therefore, it is an probabilistic protocol to simulate the time-evolution of $T$-APH two-level systems.

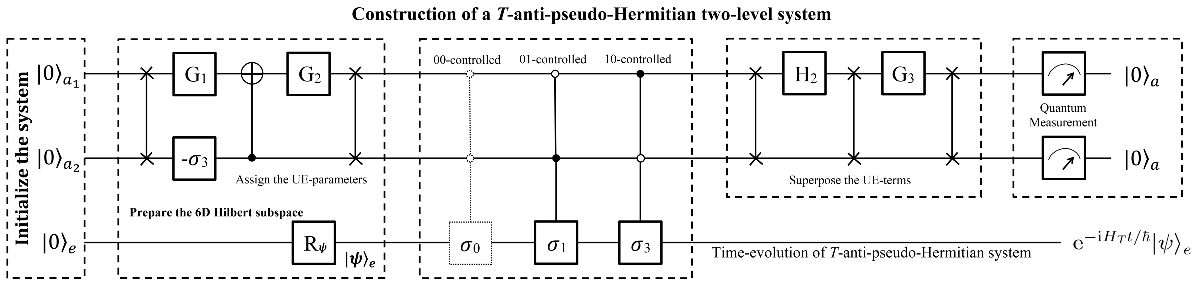


**Fig. 2.** Three-qubit quantum circuit. The system is prepared to $|00\rangle_a|\psi\rangle_e$. The six-dimensional subspace is constructed in the first block. At the meantime, the three UE-parameters are assigned. Then, the UE-terms generation is realized by three controlled-controlled operators (the first dashed one is a trivial unit matrix) in the second block. UE-terms superposition in Eq. (8) is achieved in the third block. Finally, quantum measurements are performed on the ancillary system to evolve the qubit $e$ as the $T$-APH system in a probabilistic way if $|00\rangle_a$ is obtained.

### 4.4. Qubits simulation of *PT*-APH two-level systems

The simulation of a general $PT$-anti-pseudo-Hermitian system (Fig. 3) is similar to that of a $T$-APH system, in which three qubits are essential for a general case. The operators $G_k$ ($k=1,2$) and the controlled-$\sigma_3$ in Fig. 2 should be replaced by $R_k$ ($k=1,2$) and a controlled-$\mathrm{i}\sigma_2$ as show in Fig. 3, respectively. Their explicit forms are

$$R_1=\frac{1}{h_{\mathrm{s}}}\begin{pmatrix} h_1 & -\sqrt{|h_{\mathrm{s}}|^2-|h_1|^2} \\ \sqrt{|h_{\mathrm{s}}|^2-|h_1|^2} & h_1^* \end{pmatrix}, \tag{18}$$

$$R_2=\frac{1}{\sqrt{|h_{\mathrm{s}}|^2-|h_1|^2}}\begin{pmatrix} h_2^* & h_0 \\ -h_0^* & h_2 \end{pmatrix}, \tag{19}$$

where $h_0$, $h_1$, $h_2$ and $h_{\mathrm{s}}$ are those in Eqs. (9) and (10).

The last jointly-controlled gate

$$C_{10-(\mathrm{i}\sigma_2)}=\begin{pmatrix} \sigma_0 & 0 & 0 & 0 \\ 0 & \sigma_0 & 0 & 0 \\ 0 & 0 & \mathrm{i}\sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_0 \end{pmatrix}. \tag{20}$$

Before quantum measurements, the whole system evolves to a superposition

$$\frac{1}{\sqrt{3}h_{\mathrm{s}}}\left(|00\rangle_a \mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_{PT}}|\psi\rangle_e+h_{\mathrm{s}}\sum_{k=01,10}|k\rangle_a|s_k'\rangle_e\right), \tag{21}$$

$|s_k'\rangle_e$ will be discarded after quantum measurements if $|k\rangle_a$ ($k=01,10$) is output.

At last, quantum measurements are performed on the ancillary qubits. The evolutionary qubit will probably evolve as the time-evolution $\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_{PT}}|\psi\rangle_e$ governed by the $PT$-APH Hamiltonian in Eq. (5) if $|00\rangle_a$ is measured. The success probability is

$$\frac{1}{3h_{\mathrm{s}}^2}{}_e\langle\psi|\,\mathrm{e}^{\mathrm{i}\frac{t}{\hbar}H_{PT}^\dagger}\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_{PT}}|\psi\rangle_e. \tag{22}$$

Otherwise, the simulation will be restarted until $|00\rangle_a$ is output.

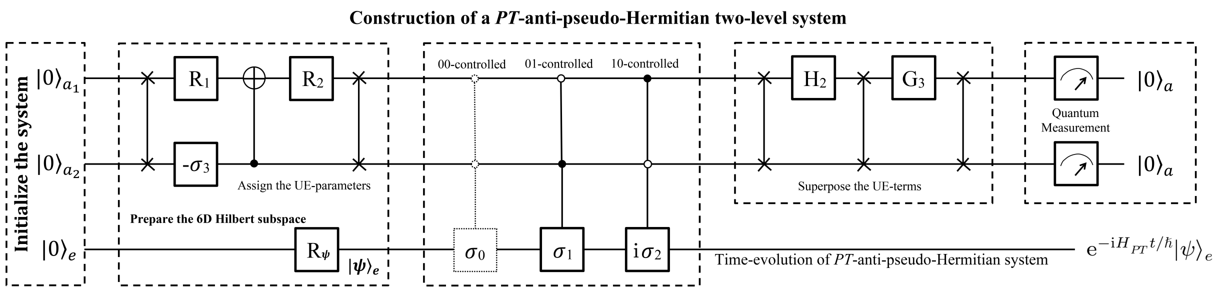


**Fig. 3.** Three-qubit quantum circuit. Detailed operations will be replaced, while the whole process is similar to that in Fig. 2. Finally, quantum measurements are performed on the ancillary subsystem to evolve the qubit $e$ as the $PT$-APH system in a probabilistic way when the output is $|00\rangle_a$.

### 4.5. Special cases

In special cases, when the Hamiltonians satisfy phase matching conditions,[6] the four UE-terms in Eq. (7) can be merged to two, and two qubits are enough to simulate the time-evolution. We take

$$H=\begin{pmatrix} u+\mathrm{i}v & s+\mathrm{i}w \\ s+\mathrm{i}w & u+\mathrm{i}v \end{pmatrix}, \tag{23}$$

as an example, which is both $T$- and $PT$-APH. The time-evolution operator can be expanded to

$$\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H}=r_0\sigma_0+r_1\sigma_1, \tag{24}$$

where the explicit forms of $r_0$ and $r_1$ are given in Appendix A (iii).

In this case, the time-evolution can be simulated by two qubits as the quantum circuit shown in Fig. 4. After prepared to $|0\rangle_a|\psi\rangle_e$, a single-qubit operator

$$V=\frac{1}{r_{\rm s}}\begin{pmatrix} r_0 & -r_1^* \\ r_1 & r_0^* \end{pmatrix} \tag{25}$$

is applied on the ancillary qubit to assign the UE-parameters, where $r_{\rm s}$ is set as $|r_0|^2+|r_1|^2$. Then,

$$C_{1-\sigma_1}=\begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_1 \end{pmatrix} \tag{26}$$

and a Hadamard $H_2$ follow. The two-qubit system evolves to

$$\frac{1}{\sqrt{2}r_{\rm s}}\left(|0\rangle_a {\rm e}^{-{\rm i}\frac{t}{\hbar}H}|\psi\rangle_e+|1\rangle_a(\sigma_0-\sigma_1)|\psi\rangle_e\right). \tag{27}$$

The first term links to the time-evolution governed by the Hamiltonian in Eq. (23), while the second term will be discarded after quantum measurements.

Finally, a quantum measurement is performed on the ancillary qubit. If $|0\rangle_a$ is measured, the evolutionary qubit $e$ will evolve as ${\rm e}^{-{\rm i}\frac{t}{\hbar}H}|\psi\rangle_e$ with a success probability of

$$\frac{1}{2r_{\rm s}^2}{}_e\langle\psi|{\rm e}^{{\rm i}\frac{t}{\hbar}H^\dagger}{\rm e}^{-{\rm i}\frac{t}{\hbar}H}|\psi\rangle_e. \tag{28}$$

Otherwise, if the ancillary qubit is observed in state $|1\rangle_a$, the process will be terminated and the result will be discarded. We will start over the quantum simulation until $|0\rangle_a$ is measured at last.

Notice that the evolution of $H$ can be simulated by both the three-qubit in the previous two subsections and this two-qubit protocol here. However, from Eqs. (17), (22) and (28), the success probability using two qubits is larger than that using three qubits (notice $r_{\rm s}$, $g_{\rm s}$ and $h_{\rm s}$ are the same for a fixed $H$). Therefore, it is meaningful to reduce UE-terms before quantum simulation, decreasing the complexities of the quantum circuit but increasing the success probability.

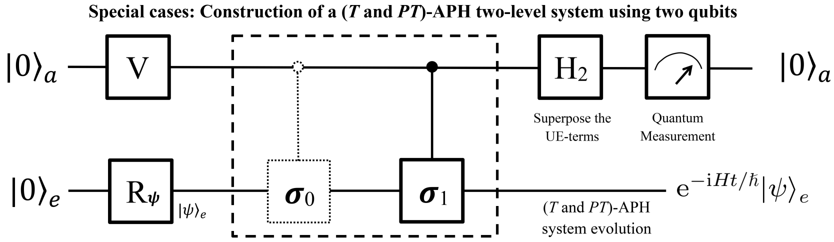


**Fig. 4.** Two-qubit quantum circuit. The system includes an ancillary qubit $a$ and an evolutionary qubit $e$. After initialized to $|0\rangle_a|\psi\rangle_e$, operators are applied in series, i.e., a single-qubit rotation $V$, two controlled operators (the first one is a trivial unit-matrix), and a Hadamard. Finally, the evolutionary qubit $e$ will evolve as ${\rm e}^{-{\rm i}\frac{t}{\hbar}H}|\psi\rangle_e$ if $|0\rangle_a$ is measured.

### 4.6. High-dimensional cases

For a high-dimensional case, assume that the dimensions of the $\tau$-APH system are $D$ and the evolutionary operator can be expanded to $N$ UE-terms. The qubit numbers of the ancillary and the evolutionary subsystems, $N_1$ and $N_2$, should not less than $\log_2 N$ and $\log_2 D$, respectively.

At the beginning, the whole system is initialized to state $|0\rangle_a|\psi\rangle_e$. Designing the quantum circuit by LCU and duality quantum algorithm, a series of operators will be applied on the system to assign the UE-parameters, generate and superpose the UE-terms to constructing the $\tau$-APH subsystem. At last, quantum measurements will be performed on the ancillary subsystem to complete the time-evolution in a probabilistic way if the ancillary qubits are measured in state $|0\rangle_a$. In this situation, the subsystem $e$ will evolve as governed by the the high-dimensional $\tau$-APH system. Otherwise, this simulation will be terminated and started over.

## 5. Discussion and conclusion

Investigations of novel phenomena by controllable and available quantum devices are one of the main tasks of quantum simulation. We discuss experimental implementations of quantum simulation of the $\tau$-APH here. Our proposals can be implemented in small quantum devices that are available technologies, such as nuclear-magnetic-resonance (NMR) quantum simulator, quantum optics systems, superconductor quantum systems, ultracold atoms, trapped ions, etc. Our proposals can also be realized on the IBM QE 5-qubit quantum processor as that in Ref. [98].

For example, the evolutionary and ancillary qubits can be realized by the nuclei of spin-$\frac{1}{2}$ in NMR. The spatial-averaging method[99] will be used to initialize the pseudo-pure state. A series of magnetic pulse sequences will be applied to realize the quantum gates in the quantum circuit. Hard pulses realize single-qubit rotations, and free evolutions of the two nuclei of spin-$\frac{1}{2}$ in a period[34] realize the relevant two-qubit controlled gates. In quantum optics, a qubit can be realized by two orthogonal polarized directions of a photon. A series of quarter-wave plates and half-wave plates[66,100] are able to realize single-qubit gates. However, the efficiency is too low in practice to realize a two-polarization-qubit gate using measurement induced nonlinearity.[101] The efficiency can be improved by introducing the location degree of freedom.[102]

In conclusion, we mainly investigate quantum simulation of the $\tau$-anti-pseudo-Hermitian systems using LCU in the scheme of duality quantum computing. In detail, we propose how to simulate the $T$-APH and $PT$-APH two-level system both in general cases using three qubits and in a special case using two qubits. In general cases, a minimum six-dimensional Hilbert space is essential to simulate the time-evolution by our unitary expansion technique. The simulation is achieved in a probabilistic way. Although the special cases can be simulated by both the two proposals, the two-qubits one has a higher success probability, deciding by the initial state, the Hamiltonian and the dimensions of the used Hilbert space. Therefore, it is meaningful to merge the UE-terms based on our UE technique and phase matching conditions to reduce the needed dimensions before quantum simulation, saving qubit source and increasing the success probability. Furthermore, we discuss schematically for general high-dimensional cases and experimental implementation of the two-dimensional cases, expecting experimental implementation in various small quantum devices.

## Appendix A

(i) Explicit forms of $g_k$'s ($k=0,1,2$):

$$g_0 = \mathrm{e}^{-\mathrm{i}\frac{t}{2\hbar}\sum_{k=1}^{2}(u_k+\mathrm{i}v_k)} \times \cos\left[\frac{t}{2\hbar}\sqrt{4(s+\mathrm{i}w)^2+\left(\sum_{k=1}^{2}(-1)^k(u_k+\mathrm{i}v_k)\right)^2}\right], \tag{A1}$$

$$g_1 = -\mathrm{i}\frac{2(s+\mathrm{i}w)}{\sqrt{4(s+\mathrm{i}w)^2+\left(\sum_{k=1}^{2}(-1)^k(u_k+\mathrm{i}v_k)\right)^2}} \times \mathrm{e}^{-\mathrm{i}\frac{t}{2\hbar}\sum_{k=1}^{2}(u_k+\mathrm{i}v_k)} \times \sin\left[\frac{t}{2\hbar}\sqrt{4(s+\mathrm{i}w)^2+\left(\sum_{k=1}^{2}(-1)^k(u_k+\mathrm{i}v_k)\right)^2}\right], \tag{A2}$$

$$g_2 = \mathrm{i}\frac{\sum_{k=1}^{2}(-1)^k(u_k+\mathrm{i}v_k)}{\sqrt{4(s+\mathrm{i}w)^2+\left(\sum_{k=1}^{2}(-1)^k(u_k+\mathrm{i}v_k)\right)^2}} \times \mathrm{e}^{-\mathrm{i}\frac{t}{2\hbar}\sum_{k=1}^{2}(u_k+\mathrm{i}v_k)} \times \sin\left[\frac{t}{2\hbar}\sqrt{4(s+\mathrm{i}w)^2+\left(\sum_{k=1}^{2}(-1)^k(u_k+\mathrm{i}v_k)\right)^2}\right]. \tag{A3}$$

(ii) Explicit forms of $h_k$'s ($k=0,1,2$):

$$h_0 = \mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}(u+\mathrm{i}v)}\cos\left[\frac{t}{\sqrt{2}\hbar}\sqrt{\sum_{k=1}^{2}(s_k+\mathrm{i}w_k)^2}\right], \tag{A4}$$

$$h_1 = -\mathrm{i}\frac{\sum_{k=1}^{2}(s_k+\mathrm{i}w_k)}{\sqrt{2\sum_{k=1}^{2}(s_k+\mathrm{i}w_k)^2}}\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}(u+\mathrm{i}v)} \times \sin\left[\frac{t}{\sqrt{2}\hbar}\sqrt{\sum_{k=1}^{2}(s_k+\mathrm{i}w_k)^2}\right], \tag{A5}$$

$$h_2 = \mathrm{i}\frac{\sum_{k=1}^{2}(-1)^k(s_k+\mathrm{i}w_k)}{\sqrt{2\sum_{k=1}^{2}(s_k+\mathrm{i}w_k)^2}}\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}(u+\mathrm{i}v)} \times \sin\left[\frac{t}{\sqrt{2}\hbar}\sqrt{\sum_{k=1}^{2}(s_k+\mathrm{i}w_k)^2}\right]. \tag{A6}$$

(iii) Explicit forms of $r_k$'s ($k=0,1$):

$$r_0 = \mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}(u+\mathrm{i}v)}\cos\frac{t}{\hbar}(s+\mathrm{i}w), \tag{A7}$$

$$r_1 = -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}(u+\mathrm{i}v)}\sin\frac{t}{\hbar}(s+\mathrm{i}w). \tag{A8}$$

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