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Detection of the quantum states containing at most $k-1$ unentangled particles*

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There are many different classifications of entanglement for multipartite quantum systems, one of which is based on the number of the unentangled particles. In this paper, we mainly study the quantum states containing at most $k-1$ unentangled particles and provide several entanglement criteria based on the different forms of inequalities, which can both identify quantum states containing at most $k-1$ unentangled particles. We show that these criteria are more effective for some states by concrete examples.

Keywords: multipartite entanglement, $k-1$ unentangled particles, Cauchy–Schwarz inequality

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1. Introduction

As a fundamental concept of quantum theory, quantum entanglement plays a crucial role in quantum information processing.[1] It has been successfully identified as a key ingredient for a wide range of applications, such as quantum cryptography,[2] quantum dense coding,[3] quantum teleportation,[4,5] factoring,[6] and quantum computation.[7,8]

One of the significant problems in the study of quantum entanglement theory is to decide whether a quantum state is entangled or not. For bipartite systems, quantum states consist of separable states and entangled states. Many well-known separability criteria have been proposed to distinguish separable from entangled states.[9,10] In multipartite case, the classification of quantum states is much more complicated due to the complex structure of multipartite quantum states. A reasonable way of classification is based on the number of partitions that are separable. According to that, N-partite quantum states can be divided into k-separable states and k-nonseparable states with $2 \leqslant k \leqslant N$. The detection of k-nonseparability has been investigated extensively, many efficient criteria[11–21] and computable measures[22–26] have been presented. Different from the above classification, N-partite quantum states can also be divided into k-producible states and $(k+1)$-partite entangled states by consideration of the number of partitions that are entangled. It is worth noting that the $(k+1)$-partite entanglement and the k-nonseparability are two different concepts involving the partitions of subsystem in N-partite quantum systems, and they are equivalent only in some special cases.

In this paper, we focus on another characterization of multipartite quantum states which is based on the number of unentangled particles. We first present the definition of quantum states containing at least k unentangled particles, and then derive several criteria to identify quantum states containing at most $k-1$ unentangled particles by using some well-known inequalities. Several specific examples illustrate the advantage of our results in detecting quantum states containing at most $k-1$ unentangled particles.

The organization of this article is as follows: In Section 2, we review the basic knowledge which will be used in the rest of the paper. In Section 3, we provide our central results, several criteria that can effectively detect quantum states containing at most $k-1$ unentangled particles, and then their strengths are exhibited by several examples. Finally, a brief summary is given in Section 4.

2. Preliminaries

In this section, we introduce the preliminary knowledge used in this paper. We consider a multiparticle quantum system with state space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, where $\mathcal{H}_i$ $(i = 1,2,\ldots,N)$ denote d_i-dimensional Hilbert spaces. For convenience, we introduce the following concepts. An N-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ contains k unentangled

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particles, if there is $k+1$ partition $\gamma_1|\gamma_2|\cdots|\gamma_{k+1}$ such that

$$|\psi\rangle=\bigotimes_{l=1}^{k+1}|\phi_l\rangle_{\gamma_l},$$

where $|\phi_l\rangle_{\gamma_l}$ is single-partite state for $1\leqslant l\leqslant k$, while $|\phi_{k+1}\rangle_{\gamma_{k+1}}$ is a $(N-k)$-particle state. A mixed state ρ contains at least k unentangled particles, if it can be written as

$$\rho=\sum_j p_j|\psi^{(j)}\rangle\langle\psi^{(j)}|,$$

where $p_j>0$ with $\sum p_j=1$, and $|\psi^{(j)}\rangle$ is the pure state containing m_j unentangled particles with $m_j\geqslant k$.[27–29] Otherwise we say ρ contains at most $k-1$ unentangled particles.

For N-partite quantum system $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$, let

$$\left\langle\bigotimes_{i=1}^{N}A_iB_i\right\rangle_\rho=\mathrm{tr}\left(\left(\bigotimes_{i=1}^{N}A_iB_i\right)\rho\right)\tag{1}$$

where ρ is the quantum state in $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$, A_i, B_i are operators acting on the i-th subsystem $\mathcal{H}_i$, and "tr" stands for trace operation.

Inequality plays an important role in quantum information theory. In the following, we list some inequalities that will be used throughout the paper.

Absolute value inequality

$$\left|\sum_{i=1}^{n}a_i\right|\leq\sum_{i=1}^{n}|a_i|.\tag{2}$$

Cauchy–Schwarz inequality

$$|\langle x|y\rangle|^2\leq\langle x|x\rangle\langle y|y\rangle,\tag{3}$$

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\leq\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right).\tag{4}$$

Extending the Cauchy–Schwarz inequality gives an important inequality known as the Hölder inequality

$$\sum_{i=1}^{n}|a_ib_i|\leq\left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|b_i|^q\right)^{\frac{1}{q}},\tag{5}$$

where $p,q>1$, and $1/p+1/q=1$.

3. Main results

Now let us state our criteria identifying quantum states containing at most $k-1$ unentangled particles for arbitrary dimensional multipartite quantum systems.

Theorem 1 If an N-partite quantum state ρ contains at least k unentangled particles for $1\leqslant k\leqslant N-1$, then it satisfies

$$\left|\langle\bigotimes_{i=1}^{N}A_iB_i\rangle_\rho\right|\leq\sum_\gamma\left(\prod_{l=1}^{k+1}\big\langle(\bigotimes_{i\in\gamma_l}A_iA_i^\dagger)\otimes(\bigotimes_{i\notin\gamma_l}B_i^\dagger B_i)\big\rangle_\rho\times\big\langle(\bigotimes_{i\in\gamma_l}B_i^\dagger B_i)\otimes(\bigotimes_{i\notin\gamma_l}A_iA_i^\dagger)\big\rangle_\rho\right)^{\frac{1}{2k+2}},\tag{6}$$

where A_i and B_i are operators acting on the i-th subsystem, and the sum runs over all possible partitions $\{\gamma|\gamma=\gamma_1|\gamma_2|\cdots|\gamma_{k+1}\}$ of N particles in which the number of particles in γ_l is 1 for $1\leq l\leq k$ and is $N-k$ for $i=k+1$. If ρ violates inequality (6), then it contains at most $k-1$ unentangled particles.

Proof Firstly, we consider the pure state $\rho=|\psi\rangle\langle\psi|$ containing k unentangled particles. Suppose that the pure state $|\psi\rangle=\bigotimes_{l=1}^{k+1}|\psi_l\rangle_{\gamma_l}$ under the partition $\gamma_1|\gamma_2|\cdots|\gamma_{k+1}$, where γ_l contains one particle for $1\leq l\leq k$, and γ_{k+1} contains $N-k$ particles. Then for any subsystems γ_l, we have

$$\begin{aligned}\left|\langle\bigotimes_{i=1}^{N}A_iB_i\rangle_\rho\right|&=\sqrt{\langle\bigotimes_{i=1}^{N}A_iB_i\rangle_\rho\langle\bigotimes_{i=1}^{N}B_i^\dagger A_i^\dagger\rangle_\rho}=\sqrt{\prod_{t=1}^{k+1}\langle\bigotimes_{i\in\gamma_t}A_iB_i\rangle_{\rho_{\gamma_t}}\prod_{t=1}^{k+1}\langle\bigotimes_{i\in\gamma_t}B_i^\dagger A_i^\dagger\rangle_{\rho_{\gamma_t}}}\leq\sqrt{\prod_{t=1}^{k+1}\langle\bigotimes_{i\in\gamma_t}A_iA_i^\dagger\rangle_{\rho_{\gamma_t}}\prod_{t=1}^{k+1}\langle\bigotimes_{i\in\gamma_t}B_i^\dagger B_i\rangle_{\rho_{\gamma_t}}}\\&=\sqrt{\langle\bigotimes_{i\in\gamma_l}A_iA_i^\dagger\rangle_{\rho_{\gamma_l}}\prod_{t\neq l}\langle\bigotimes_{i\in\gamma_t}A_iA_i^\dagger\rangle_{\rho_{\gamma_t}}\langle\bigotimes_{i\in\gamma_l}B_i^\dagger B_i\rangle_{\rho_{\gamma_l}}\prod_{t\neq l}\langle\bigotimes_{i\in\gamma_t}B_i^\dagger B_i\rangle_{\rho_{\gamma_t}}}\\&=\sqrt{\langle(\bigotimes_{i\in\gamma_l}A_iA_i^\dagger)\otimes(\bigotimes_{i\notin\gamma_l}B_i^\dagger B_i)\rangle_\rho\langle(\bigotimes_{i\in\gamma_l}B_i^\dagger B_i)\otimes(\bigotimes_{i\notin\gamma_l}A_iA_i^\dagger)\rangle_\rho},\end{aligned}$$

where $\rho_{\gamma_t}=|\psi_t\rangle_{\gamma_t}\langle\psi_t|$. Here we have used the Cauchy–Schwarz inequality (3). Thus,

$$\begin{aligned}\left|\langle\bigotimes_{i=1}^{N}A_iB_i\rangle_\rho\right|&\leq\left(\prod_{l=1}^{k+1}\sqrt{\big\langle(\bigotimes_{i\in\gamma_l}A_iA_i^\dagger)\otimes(\bigotimes_{i\notin\gamma_l}B_i^\dagger B_i)\big\rangle_\rho\big\langle(\bigotimes_{i\in\gamma_l}B_i^\dagger B_i)\otimes(\bigotimes_{i\notin\gamma_l}A_iA_i^\dagger)\big\rangle_\rho}\right)^{\frac{1}{k+1}}\\&\leq\sum_\gamma\left(\prod_{l=1}^{k+1}\sqrt{\big\langle(\bigotimes_{i\in\gamma_l}A_iA_i^\dagger)\otimes(\bigotimes_{i\notin\gamma_l}B_i^\dagger B_i)\big\rangle_\rho\big\langle(\bigotimes_{i\in\gamma_l}B_i^\dagger B_i)\otimes(\bigotimes_{i\notin\gamma_l}A_iA_i^\dagger)\big\rangle_\rho}\right)^{\frac{1}{k+1}}\\&=\sum_\gamma\left(\prod_{l=1}^{k+1}\big\langle(\bigotimes_{i\in\gamma_l}A_iA_i^\dagger)\otimes(\bigotimes_{i\notin\gamma_l}B_i^\dagger B_i)\big\rangle_\rho\big\langle(\bigotimes_{i\in\gamma_l}B_i^\dagger B_i)\otimes(\bigotimes_{i\notin\gamma_l}A_iA_i^\dagger)\big\rangle_\rho\right)^{\frac{1}{2k+2}}.\end{aligned}$$

It shows that inequality (6) is right for pure state containing k unentangled particles.

Now, we consider the case of the mixed state. Suppose $\rho=\sum_j p_j\rho_j$ is a mixed state with pure states ρ_j containing at least k unentangled particles, then

$$\begin{aligned}
&\left|\langle\bigotimes_{i=1}^{N}A_iB_i\rangle_\rho\right| \le \sum_j p_j\left|\langle\bigotimes_{i=1}^{N}A_iB_i\rangle_{\rho_j}\right| \\
&\le \sum_j p_j\sum_\gamma\Bigg(\prod_{l=1}^{k+1}\big\langle(\bigotimes_{i\in\gamma_l}A_iA_i^\dagger)\otimes(\bigotimes_{i\notin\gamma_l}B_i^\dagger B_i)\big\rangle_{\rho_j} \\
&\quad\times\big\langle(\bigotimes_{i\in\gamma_l}B_i^\dagger B_i)\otimes(\bigotimes_{i\notin\gamma_l}A_iA_i^\dagger)\big\rangle_{\rho_j}\Bigg)^{\frac{1}{2k+2}} \\
&\le \sum_\gamma\Bigg(\prod_{l=1}^{k+1}\sum_j p_j\big\langle(\bigotimes_{i\in\gamma_l}A_iA_i^\dagger)\otimes(\bigotimes_{i\notin\gamma_l}B_i^\dagger B_i)\big\rangle_{\rho_j}^{\frac{1}{2}} \\
&\quad\times\big\langle(\bigotimes_{i\in\gamma_l}B_i^\dagger B_i)\otimes(\bigotimes_{i\notin\gamma_l}A_iA_i^\dagger)\big\rangle_{\rho_j}^{\frac{1}{2}}\Bigg)^{\frac{1}{k+1}} \\
&\le \sum_\gamma\Bigg(\prod_{l=1}^{k+1}\big\langle(\bigotimes_{i\in\gamma_l}A_iA_i^\dagger)\otimes(\bigotimes_{i\notin\gamma_l}B_i^\dagger B_i)\big\rangle_{\rho} \\
&\quad\times\big\langle(\bigotimes_{i\in\gamma_l}B_i^\dagger B_i)\otimes(\bigotimes_{i\notin\gamma_l}A_iA_i^\dagger)\big\rangle_{\rho}\Bigg)^{\frac{1}{2k+2}},
\end{aligned}$$

where we have used the absolute value inequality (2), inequality (6) for pure states, the Hölder inequality (5) and Cauchy–Schwarz inequality (4). The proof is complete.

Theorem 2 For any N-partite density matrix acting on Hilbert space $\rho\in\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ containing at least k unentangled particles, where $1\le k\le N-2$, we have

$$\begin{aligned}
&\sum_{m\neq n}\left|\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho\right| \\
&\le \sum_{m\neq n}\sqrt{\langle\bigotimes_{i=1}^{N}A_iA_i^\dagger\rangle_\rho\langle U_mU_n(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger U_m^\dagger\rangle_\rho} \\
&\quad+(N-k-1)\sum_m\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_m^\dagger\rangle_\rho, \qquad (7)
\end{aligned}$$

where A_i is any operator of the subsystem $\mathcal{H}_i$, and $U_m=\mathbf{1}_1\otimes\cdots\otimes\mathbf{1}_{m-1}\otimes u_m\otimes\mathbf{1}_{m+1}\otimes\cdots\otimes\mathbf{1}_N$ with u_m being any operator of the subsystem $\mathcal{H}_m$ and $\mathbf{1}_j$ being identity matrix of the subsystem $\mathcal{H}_j$. If ρ violates inequality (7), then it contains at most $k-1$ unentangled particles.

Proof We begin with the pure state. Suppose that the pure state $|\psi\rangle$ contains k unentangled particles, then there is a partition $\gamma_1|\cdots|\gamma_{k+1}$ with γ_l containing one particle for $1\le l\le k$, and γ_{k+1} containing $N-k$ particles, and it can be written as $|\psi\rangle=\bigotimes_{l=1}^{k+1}|\psi_l\rangle_{\gamma_l}$. When m,n in γ_{k+1}, we can obtain

$$\begin{aligned}
&\left|\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho\right| \\
&\le \sqrt{\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_m^\dagger\rangle_\rho\langle U_n(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho} \\
&\le \frac{\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_m^\dagger\rangle_\rho+\langle U_n(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho}{2}, \qquad (8)
\end{aligned}$$

where the first inequality holds because of Cauchy–Schwarz inequality (3) and the second inequality follows from the mean inequality. When $m\in\gamma_l$, $n\in\gamma_{l'}$ and $l\neq l'$, we have

$$\left|\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho\right|\le\sqrt{\langle(\bigotimes_{i=1}^{N}A_iA_i^\dagger)\rangle_\rho\langle U_mU_n(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger U_m^\dagger\rangle_\rho} \qquad (9)$$

by Cauchy–Schwarz inequality (3).

Combining inequalities (8) and (9) leads to

$$\begin{aligned}
&\sum_{m\neq n}\left|\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho\right| \\
&=\sum_{m\in\gamma_l,n\in\gamma_{l'},l\neq l'}\left|\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho\right| \\
&\quad+\sum_{m,n\in\gamma_{k+1},m\neq n}\left|\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho\right| \\
&\le\sum_{m\in\gamma_l,n\in\gamma_{l'},l\neq l'}\sqrt{\langle(\bigotimes_{i=1}^{N}A_iA_i^\dagger)\rangle_\rho\langle U_mU_n(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger U_m^\dagger\rangle_\rho} \\
&\quad+\frac{1}{2}\sum_{m,n\in\gamma_l,m\neq n}\left(\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_m^\dagger\rangle_\rho+\langle U_n(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho\right) \\
&\le\sum_{m\neq n}\sqrt{\langle\bigotimes_{i=1}^{N}A_iA_i^\dagger\rangle_\rho\langle U_mU_n(\bigotimes_{i=1}^{n}A_iA_i^\dagger)U_n^\dagger U_m^\dagger\rangle_\rho} \\
&\quad+(N-k-1)\sum_m\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_m^\dagger\rangle_\rho.
\end{aligned}$$

Hence, inequality (7) holds for any pure state containing k unentangled particles. It is easy to prove that it is also right for any mixed state containing at least k unentangled particles by utilizing absolute value inequality, inequality (7) for pure states, and the Cauchy–Schwarz inequality (4).

Theorem 3 For any N-partite fully separable state ρ, one has

$$\left|\langle U_m(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger\rangle_\rho\right|\le\sqrt{\langle(\bigotimes_{i=1}^{N}A_iA_i^\dagger)\rangle_\rho\langle U_mU_n(\bigotimes_{i=1}^{N}A_iA_i^\dagger)U_n^\dagger U_m^\dagger\rangle_\rho} \qquad (10)$$

for any $m\neq n$. If ρ does not satisfy the above inequality (10), then it is entangled.

Proof The proof of this result is quite similar to Theorem 2. Note that there is only one case that m,n belong to different γ_l if $\rho=|\psi\rangle\langle\psi|$ is fully separable pure state, which ensures that inequality (10) is true for fully separable pure state. Hence, inequality (10) also holds for fully separable mixed states.

The following examples shows that the power of our results by comparison with observation 5 in Ref. [29].

Example 1 For the family of quantum states

$$\rho(p) = p|\Psi_5\rangle\langle\Psi_5| + \frac{1-p}{5^5}\mathbf{1},$$

where $|\Psi_5\rangle = \frac{1}{\sqrt{5}}\sum_{i=0}^{4}|iiiii\rangle$.

Applying Theorem 1 by choosing $A_i = |1\rangle\langle 0|$ and $B_i = |0\rangle\langle 0|$, we can get that, if $p > 0.0016$, ρ contains at most 3 unentangled particles; if $p > 0.0173$, ρ contains at most 2 unentangled particles; if $p > 0.0325$, ρ contains at most 1 unentangled particles; and if $p > 0.0399$, ρ contains at most 0 unentangled particles. But observation 5 in Ref. [29] cannot detect any quantum states containing at most k unentangled particles for $0 \le k \le 3$.

Example 2 Consider the N-qubit mixed states

$$\rho(p) = p|G\rangle\langle G| + \frac{1-p}{2^N}\mathbf{1},$$

where $|G\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$.

Let $A_i = |1\rangle\langle 0|, B_i = |0\rangle\langle 0|$, then by using Theorem 1, we know that $\rho(p)$ contains at most $N-3$ unentangled particles when $p_{N-2} < p \le 1$, while by observation 5 in Ref. [29], $\rho(p)$ contains at most $N-3$ unentangled particles when $p'_{N-2} < p \le 1$. The exact value of p_{N-2} and p'_{N-2} for N = 9,10,...,15 are shown in Table 1.

Table 1. For $\rho(p) = p|G\rangle\langle G| + \frac{1-p}{2^N}\mathbf{1}$, the thresholds of p_{N-2}, p'_{N-2} for the quantum states containing at most $N-3$ unentangled particles detected by Theorem 1 and observation 5 in Ref. [29] for $9 \le N \le 15$, respectively, are illustrated. When $p_{N-2} < p \le 1$ and $p'_{N-2} < p \le 1$, $\rho(p)$ contains at most $N-3$ unentangled particles by Theorem 1 and observation 5 in Ref. [29], respectively. Clearly, Theorem 1 can detect more states containing at most $N-3$ unentangled particles than observation 5 in Ref. [29] for $9 \le N \le 15$.

N	9	10	11	12	13	14	15
p_{N-2}	0.1263	0.0824	0.0519	0.0317	0.0189	0.0111	0.0064
p'_{N-2}	0.1547	0.1350	0.1197	0.1076	0.0977	0.0894	0.0824

Example 3 Consider the N-qubit mixed states

$$\rho(p,q) = p|W_N\rangle\langle W_N| + q\sigma_x^{\otimes N}|W_N\rangle\langle W_N|\sigma_x^{\otimes N} + \frac{1-p-q}{2^N}\mathbf{1}.$$

Here $|W_N\rangle = \frac{1}{\sqrt{N}}(|10\cdots 0\rangle + |01\cdots 0\rangle + \cdots + |0\cdots 01\rangle)$ and σ_x is the Pauli matrix.

By choosing $u_m = \sigma_x$, $A_i = |0\rangle\langle 0|$ (or $A_i = |1\rangle\langle 0|$), our Theorem 2 can identify quantum states containing at most $k-1$ unentangled particles. For $k = 2$, the detection parameter spaces in which the quantum states contains at most 1 unentangled particles when $N = 6,7,8,9$ are shown in Fig. 1.

When $p = 0$, the quantum state $\rho(p,q)$ is

$$\rho(q) = q\sigma_x^{\otimes N}|W_N\rangle\langle W_N|\sigma_x^{\otimes N} + \frac{1-q}{2^N}\mathbf{1}.$$

By choosing $u_m = \sigma_x$, $A_i = |1\rangle\langle 0|$, our Theorem 2 can identify more quantum states containing at most $k-1$ unentangled particles than observation 5 in Ref. [29] when $N = 8$. For $1 \le k \le 6$, when $q_k < q \le q'_k$, these quantum states containing at most $k-1$ unentangled particles which only can be detected by our Theorem 2, but not by observation 5 in Ref. [29]. The exact values of q_k and q'_k for $1 \le k \le 6$ are shown in Table 2.

Table 2. For $\rho(q) = q\sigma_x^{\otimes N}|W_N\rangle\langle W_N|\sigma_x^{\otimes N} + \frac{1-q}{2^N}\mathbf{1}$ when $N = 8$, the thresholds of q_k, q'_k for the quantum states containing at most $k-1$ unentangled particles detected by Theorem 2 and observation 5 in Ref. [29] for $1 \le k \le 6$, respectively, are illustrated. When $q_k < q \le 1$ and $q'_k < q \le 1$, $\rho(q)$ contains at most $k-1$ unentangled particles detected by our Theorem 2 and observation 5 in Ref. [29], respectively. The symbol \ means that observation 5 in Ref. [29] cannot identify any quantum states containing at most 0 unentangled particles and at most 1 unentangled particles.

k	1	2	3	4	5	6
q_k	0.2889	0.1579	0.1028	0.0725	0.0533	0.0400
q'_k	\	\	0.8647	0.6392	0.4587	0.3234

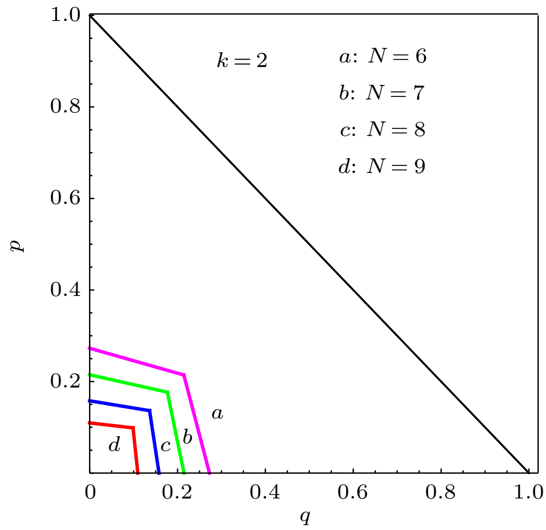

Fig. 1. Detection quality of Theorem 2 for the state $\rho(p,q) = p|W_N\rangle\langle W_N| + q\sigma_x^{\otimes N}|W_N\rangle\langle W_N|\sigma_x^{\otimes N} + \frac{1-p-q}{2^N}\mathbf{1}$ for $k = 2$ when $N = 6,7,8,9$. The area enclosed by magenta a (green line b, blue line c, red line d), p axis, line $q = 1-p$ and q axis corresponds to the quantum states containing at most 1 unentangled particles when $N = 6$ ($N = 7$, $N = 8$, $N = 9$), respectively.

4. Conclusion

In this paper, we have investigated the problem of detection of quantum states containing at most $k-1$ unentangled particles. Several criteria for detecting states containing at most $k-1$ unentangled particles were presented for arbitrary dimensional multipartite quantum systems. It turned out that our results were effective by some specific examples. We hope that our results can contribute to a further understanding of entanglement properties of multipartite quantum systems.

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