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Solutions to Nonlocal Integrable Discrete Nonlinear Schrödinger Equations via Reduction *

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Solutions to local and nonlocal integrable discrete nonlinear Schrödinger (IDNLS) equations are studied via reduction on the bilinear form. It is shown that these solutions to IDNLS equations can be expressed in terms of the single Casorati determinant under different constraint conditions.

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Recently, the nonlocal nonlinear integrable system has attracted considerable attention after Ablowitz and Musslimani found the parity-time (PT) symmetry nonlinear Schrödinger (NLS) equation from a nonlocal reduction of the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem.[1,2] Many nonlocal integrable equations have been established from the general AKNS scattering problem via the different symmetry reductions involving the reverse space-time symmetry, or the partially PT symmetry and the partially reverse space-time symmetry in the higher dimensional case.[1−9] These nonlocal systems have potential application in nonlinear PT-symmetry media[4] and more universal Alice-Bob events.[5,6] Due to the integrability of such nonlocal models, they can be treated by classical methods such as inverse scattering transform, Darboux transformation and bilinear approach.[10−25]

As integrable discrete analogues of local and nonlocal NLS equations, there exist four reductions from the Ablowitz–Ladik (AL) spectral problem[2,10,23]

$$
\begin{aligned}
i\psi_{n,t} &= \psi_{n+1} + \psi_{n-1} - 2\psi_n \\
&\quad - \delta\psi_n\psi_n^*(\psi_{n+1} + \psi_{n-1}), \qquad (1)\\
i\psi_{n,t} &= \psi_{n+1} + \psi_{n-1} - 2\psi_n \\
&\quad - \delta\psi_n\psi_{-n}^*(\psi_{n+1} + \psi_{n-1}), \qquad (2)\\
i\psi_{n,t} &= \psi_{n+1} + \psi_{n-1} - 2\psi_n \\
&\quad - \gamma\psi_n\psi_n(-t)(\psi_{n+1} + \psi_{n-1}), \qquad (3)\\
i\psi_{n,t} &= \psi_{n+1} + \psi_{n-1} - 2\psi_n \\
&\quad - \gamma\psi_n\psi_{-n}(-t)(\psi_{n+1} + \psi_{n-1}), \qquad (4)
\end{aligned}
$$

where $\delta = \pm 1$, and γ is an arbitrary complex constant. The four cases correspond to the standard AL symmetry, the discrete PT preserved symmetry, the reverse time discrete symmetry and the reverse discrete-time symmetry.[2,10,11,17,21,23] Very recently, a bilinearisation-reduction approach has been used to derive solutions to Eqs. (1) and (4) uniformly and these solutions were expressed in terms of double Casorati determinant.[23] It is noted that for the local integrable discrete NLS (IDNLS) Eq. (1), its bright soliton solutions can be expressed by the double Casorati determinant whereas its dark soliton solutions are given by the single Casorati determinant.[26] Therefore, the question arises whether nonlocal IDNLS equations allow the single Casorati determinant solutions or not. To answer this question, our goal in the present study is to derive general single Casorati determinant solutions to the defocusing local Eq. (1), the PT-symmetry Eq. (2) and the reverse discrete-time symmetric Eq. (4). In this Letter, we firstly introduce the before-reduction IDNLS equation whose solution is expressed in terms of the single Casorati determinant. Then solutions to local and nonlocal IDNLS equations are derived by imposing different constraint conditions.

Let us firstly recall the Casorati solution of the before-reduction IDNLS equation. According to the derivation in Ref. [26], the before-reduction IDNLS equations

$$
\begin{aligned}
& i\frac{du_n}{dt} + (c+d)u_n - [ab - (ab-1)u_nv_n] \\
& \cdot (du_{n+1} + cu_{n-1}) = 0, \qquad (5)\\
& -i\frac{dv_n}{dt} + (c+d)v_n - [ab - (ab-1)u_nv_n] \\
& \cdot (cv_{n+1} + dv_{n-1}) = 0 \qquad (6)
\end{aligned}
$$

can be transformed into the bilinear form

$$
(iD_t + c + d)g_n \cdot f_n - dg_{n+1}f_{n-1} - cg_{n-1}f_{n+1} = 0, \qquad (7)
$$

$$
(iD_t - c - d)h_n \cdot f_n + ch_{n+1}f_{n-1} + dh_{n-1}f_{n+1} = 0, \qquad (8)
$$

$$
f_{n+1}f_{n-1} - f_n^2 = (ab-1)(f_n^2 - g_nh_n), \qquad (9)
$$

via the dependent variable transformation $u_n = g_n/f_n$ and $v_n = h_n/f_n$.

Through the dimensional reduction from bilinear equations of the Bäcklund transformation of Toda lattice and the discrete two-dimensional Toda lattice, it is found that the bilinear IDNLS Eqs. (7)–(9) possess the solution in terms of Casorati determinant

$$
f_n = |F_{N\times N}| = |p_i^{n+j-1}e^{\xi_i} + q_i^{n+j-1}e^{\eta_i}|_{N\times N}, \qquad (10)
$$

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$$g_n=|G_{N\times N}|=\Big|\frac{p_i^{n+j}}{1-ap_i}e^{\xi_i}+\frac{q_i^{n+j}}{1-aq_i}e^{\eta_i}\Big|_{N\times N},\tag{11}$$

$$h_n=|H_{N\times N}|=\Big|\frac{p_i^{n+j-2}}{(1-ap_i)^{-1}}e^{\xi_i}+\frac{q_i^{n+j-2}}{(1-aq_i)^{-1}}e^{\eta_i}\Big|_{N\times N},\tag{12}$$

with $\xi_i=i(acp_i+\frac{bd}{p_i})t+\xi_{i,0}$ and $\eta_i=i(acq_i+\frac{bd}{q_i})t+\eta_{i,0}$, where the wave numbers p_i and q_i need to satisfy the constraint condition

$$q_i=-p_i\frac{1-b\frac{1}{p_i}}{1-ap_i}.\tag{13}$$

In the following, we consider reductions for local and nonlocal IDNLS equations.

(A) The defocusing local IDNLS Eq. (1): Firstly, we introduce the following diagonal matrices and a Vandermonde matrix

$$\begin{aligned}A&=\mathrm{Diag}(a_1,a_2,\ldots,a_N),\\ B&=\mathrm{Diag}(b_1,b_2,\ldots,b_N),\\ C&=\mathrm{Diag}(c_1,c_2,\ldots,c_N),\end{aligned}\tag{14}$$

$$V_q=\Big[(-1)^{N-i}\sum_{\substack{1\le k_1<k_2<\cdots<k_{N-i}\le N\\ k_l\neq j}}\Big(\prod_{l=1}^{N-i}q_{k_l}\Big)\Big]_{N\times N},\tag{15}$$

with the elements $a_i=(-1)^{N+1}\frac{p_i}{p_i-q_i}$, $b_i=\frac{1}{q_i^n e^{\eta_i'}}$ and $c_i=\frac{1-ap_i}{p_i}$. For example, when $N=1,2,3$, the Vandermonde matrices read

$$V_q=(\,1\,),\ V_q=\begin{pmatrix}-q_2&-q_1\\1&1\end{pmatrix},$$

$$V_q=\begin{pmatrix}q_2q_3&q_3q_1&q_1q_2\\-q_2-q_3&-q_3-q_1&-q_1-q_2\\1&1&1\end{pmatrix}.$$

Meanwhile, we take exponential functions in Casorati determinants as

$$\exp(\xi_i)=[\prod_{\substack{k=1\\k\neq i}}^{N}(q_k-p_i)]^{-1}\exp(\xi_i'),$$
$$\exp(\eta_i)=[\prod_{\substack{k=1\\k\neq i}}^{N}(q_k-q_i)]^{-1}\exp(\eta_i'),\tag{16}$$

with $\xi_i'=i(acp_i+\frac{bd}{p_i})t+\xi_{i,0}'$ and $\eta_i'=i(acq_i+\frac{bd}{q_i})t+\eta_{i,0}'$.

Due to the gauge freedom, we can find that

$$\begin{aligned}\tilde f_n&=|\tilde F|=|AFV_qB|\\&=\Big|\frac{\delta_{ij}}{1-\frac{q_i}{p_i}}+\frac{1}{1-\frac{q_j}{p_i}}\frac{p_i^ne^{\xi_i'}}{q_j^ne^{\eta_j'}}\Big|_{N\times N},\end{aligned}\tag{17}$$

$$\begin{aligned}\tilde g_n&=|\tilde G|=|CAGV_qB|\\&=\Big|\frac{\delta_{ij}}{1-\frac{q_i}{p_i}}\frac{\frac{1}{p_i}-a}{\frac{1}{q_i}-a}+\frac{1}{1-\frac{q_j}{p_i}}\frac{p_i^ne^{\xi_i'}}{q_j^ne^{\eta_j'}}\Big|_{N\times N},\end{aligned}\tag{18}$$

$$\begin{aligned}\tilde h_n&=|\tilde H|=|C^{-1}AHV_qB|\\&=\Big|\frac{\delta_{ij}}{1-\frac{q_i}{p_i}}\frac{\frac{1}{q_i}-a}{\frac{1}{p_i}-a}+\frac{1}{1-\frac{q_j}{p_i}}\frac{p_i^ne^{\xi_i'}}{q_j^ne^{\eta_j'}}\Big|_{N\times N},\end{aligned}\tag{19}$$

still satisfy the bilinear IDNLS Eqs. (7)–(9).

To conduct the complex conjugate reduction, we impose the constraints

$$\begin{aligned}&q_i=\frac{1}{p_i^*},\quad a^*c^*=bd,\\&\xi_{i,0}^{'*}=-\eta_{i,0}^{'},\quad \Big|\frac{1}{p_i}-a\Big|^2=|p_i-a^*|^2.\end{aligned}\tag{20}$$

Then, we have the relations

$$\tilde F_{ij}^*=\tilde F_{ji},\quad \tilde G_{ij}^*=\tilde H_{ji},\tag{21}$$

which means $\tilde f_n^*=\tilde f_n$ and $\tilde g_n^*=\tilde h_n$. Thus it immediately gives $u_n^*=\tilde g_n^*/\tilde f_n^*=\tilde h_n/\tilde f_n=v_n$.

Moreover, a direct substitution to the constraint condition (13) yields

$$\frac{1}{p_i}-a=-\frac{p_i^*}{p_i}(p_i-b),\quad\text{or}\quad |\frac{1}{p_i}-a|^2=|p_i-b|^2.\tag{22}$$

Comparing the conditions (20) and (22), we simply take $b=a^*$ and $d=c^*$ to ensure the dimensional constraint condition (13) held.

In addition, if we let $p_i=k_i+ih_i$ and $a=a_1+ia_2$, then the condition $|\frac{1}{p_i}-a|^2=|p_i-a^*|^2$ gives rise to $h_i=-a_2\pm\sqrt{2a_1k_i+a_2^2-k_i^2-1}$. It requires $2a_1k_i+a_2^2-k_i^2-1>0$, which means that the discriminant $4(a_1^2+a_2^2-1)$ needs to be positive for k_i. That is to say, the parameter a needs to satisfy $|a|^2>1$.

Therefore, the before-reduction IDNLS Eqs. (5) and (6) reduce to the single equation with $v_n^*=u_n$, $b=a^*$, $d=c^*$ and $|a|^2>1$. Through the variable transformations

$$\begin{aligned}&u_n=\frac{\psi_n}{\sqrt{\alpha}}\exp\Big(-in\theta+i\frac{e^{i\theta}+e^{-i\theta}}{|a|^2}t-2it\Big),\\&\alpha=\frac{|a|^2-1}{|a|^2},\qquad c=\frac{e^{-i\theta}}{|a|^2},\end{aligned}\tag{23}$$

one can obtain the defocusing local IDNLS Eq. (1) with $\delta=1$. Finally, we obtain the following theorem about the dark soliton solution of the IDNLS equation, which coincides with the result in Ref. [26].

Theorem 1.1: The defocusing local IDNLS Eq. (1) has the dark soliton solution

$$\psi_n=\sqrt{\frac{|a|^2-1}{|a|^2}}e^{(in\theta-i\frac{e^{i\theta}+e^{-i\theta}}{|a|^2}t+2it)}\frac{\tilde g_n}{\tilde f_n},\tag{24}$$

where

$$\tilde f_n=\Big|\frac{\delta_{ij}}{1-\frac{1}{|p_i|^2}}+\frac{1}{1-\frac{1}{p_ip_j^*}}p_i^np_j^{*n}e^{\xi_i'+\xi_j^{'*}}\Big|_{N\times N},$$

$$\tilde g_n=\Big|\frac{\delta_{ij}}{1-\frac{1}{|p_i|^2}}\frac{\frac{1}{p_i}-a}{p_i^*-a}+\frac{1}{1-\frac{1}{p_ip_j^*}}p_i^np_j^{*n}e^{\xi_i'+\xi_j^{'*}}\Big|_{N\times N},\tag{25}$$

with $\xi_i'=i(\frac{ap_i}{e^{i\theta}}+\frac{a^*e^{i\theta}}{p_i})\frac{t}{|a|^2}+\xi_{i,0}'$ and the parameters need to satisfy the constraint condition

$$\frac{1}{p_i}-a=-\frac{p_i^*}{p_i}(p_i-a^*). \tag{26}$$

For example, the defocusing local IDNLS Eq. (1) possesses a one-soliton solution

$$\psi_n=\sqrt{\frac{|a|^2-1}{|a|^2}}e^{(in\theta-i\frac{e^{i\theta}+e^{-i\theta}}{|a|^2}t+2it)}\cdot\frac{\frac{1-ap_1}{p_1(p_i^*-a)}+|p_1|^{2n}e^{\xi_1'+\xi_1'^*}}{1+|p_1|^{2n}e^{\xi_1'+\xi_1'^*}}, \tag{27}$$

with $\xi_1'=i(\frac{ap_1}{e^{i\theta}}+\frac{a^*e^{i\theta}}{p_1})\frac{t}{|a|^2}+\xi_{1,0}'$, where a, p_1 and $\xi_{1,0}'$ are arbitrary complex parameters, θ is an arbitrary real parameter and these parameters need to satisfy $\frac{1}{p_1}-a=-\frac{p_1^*}{p_1}(p_i-a^*)$ and $|a|^2>1$.

(B) The PT-symmetry IDNLS Eq. (1): For the second nonlocal IDNLS, we take the same diagonal matrices B and C and the same Vandermonde matrix V_q in Eq. (14) except for the elements of matrix A which is replaced by $a_i=(-1)^{N+1}\frac{1}{p_i-q_i}$. The exponential functions in Casorati determinants are also changed the same as Eq. (16). Owing to the gauge freedom, one can find that

$$\begin{aligned}\tilde f_n&=|\tilde F|=|AFV_qB|\\&=\Big|\frac{\delta_{ij}}{p_i-q_i}+\frac{1}{p_i-q_j}\frac{p_i^ne^{\xi_i'}}{q_j^ne^{\eta_j'}}\Big|_{N\times N},\end{aligned} \tag{28}$$

$$\begin{aligned}\tilde g_n&=|\tilde G|=|CAGV_qB|\\&=\Big|\frac{\delta_{ij}}{p_i-q_i}\frac{\frac{1}{p_i}-a}{\frac{1}{q_i}-a}+\frac{1}{p_i-q_j}\frac{p_i^ne^{\xi_i'}}{q_j^ne^{\eta_j'}}\Big|_{N\times N},\end{aligned} \tag{29}$$

$$\begin{aligned}\tilde h_n&=|\tilde H|=|C^{-1}AHV_qB|\\&=\Big|\frac{\delta_{ij}}{p_i-q_i}\frac{\frac{1}{q_i}-a}{\frac{1}{p_i}-a}+\frac{1}{p_i-q_j}\frac{p_i^ne^{\xi_i'}}{q_j^ne^{\eta_j'}}\Big|_{N\times N}\end{aligned} \tag{30}$$

are still the solution to the bilinear IDNLS Eqs. (7)–(9).

If we impose the following conditions

$$\begin{aligned}&q_i=p_i^*,\quad a^*c^*=ac,\quad b^*d^*=bd,\\&\xi_{i,0}'^*=-\eta_{i,0}',\quad |1-ap_i|^2=|1-a^*p_i|^2,\end{aligned} \tag{31}$$

then we have

$$\tilde F_{ij}^*(-n)=-\tilde F_{ji}(n),\quad \tilde G_{ij}^*(-n)=-\tilde H_{ji}(n), \tag{32}$$

which leads to $\tilde f_{-n}^*=(-1)^N\tilde f_n$ and $\tilde g_{-n}^*=(-1)^N\tilde h_n$. Thus it immediately reaches $u_{-n}^*=\frac{\tilde g_{-n}^*}{\tilde f_{-n}^*}=\frac{\tilde h_n}{\tilde f_n}=v_n$.

Furthermore, a direct substitution to the constraint condition (13) yields

$$1-ap_i=-\frac{p_i}{p_i^*}\Big(1-\frac{b}{p_i}\Big),\ \text{ or }\ |1-ap_i|^2=\Big|1-\frac{b}{p_i}\Big|^2. \tag{33}$$

If both the equations in the before-reduction IDNLS Eq. (5) and (6) are consistent, we will require $a^*=a$, $b^*=b$, $c^*=c$ and $d^*=d$.

Similarly, let $p_i=k_i+ih_i$, then the constraint condition $|1-ap_i|^2=|1-\frac{b}{p_i}|^2$ generates $h_i=\pm\frac{1}{a}\sqrt{-a^2k_i^2+2ak_i-ab}$. It requires $-a^2k_i^2+2ak_i-ab>0$, which suggests that the discriminant $1-ab$ needs to be positive for k_i. That is to say, the parameters a and b need to satisfy the condition $ab<1$.

Therefore, the before-reduction IDNLS Eqs. (5) and (6) become the single equation ($v_{-n}^*=u_n$ and $ab<1$) with the PT symmetry invariance. Through the variable transformations

$$\begin{aligned}&u_n=\frac{\psi_n}{\sqrt{\alpha}}\exp\Big(-n\theta+i\frac{e^\theta+e^{-\theta}}{ab}t-2it\Big),\\&\alpha=\delta\frac{ab-1}{ab},\quad c=\frac{e^{-\theta}}{ab},\quad d=\frac{e^\theta}{ab},\end{aligned} \tag{34}$$

with $\delta=1$ $(ab<0)$ and $\delta=-1$ $(0<ab<1)$, we obtain the PT-symmetry IDNLS Eq. (1). Finally, we arrive at the following theorem about the solution of the nonlocal IDNLS Eq. (1).

Theorem 1.2: The PT-symmetry IDNLS Eq. (1) has the solution

$$\psi_n=\sqrt{\frac{ab-1}{\delta ab}}e^{(n\theta-i\frac{e^\theta+e^{-\theta}}{ab}t+2it)}\frac{\tilde g_n}{\tilde f_n}, \tag{35}$$

with $\delta=1(ab<0)$ and $\delta=-1(0<ab<1)$, where

$$\begin{aligned}\tilde f_n&=\Big|\frac{\delta_{ij}}{p_i-p_i^*}+\frac{1}{p_i-p_j^*}\frac{p_i^n}{p_j^{*n}}e^{\xi_i'+\xi_j'^*}\Big|_{N\times N},\\ \tilde g_n&=\Big|\frac{\delta_{ij}}{p_i-p_i^*}\frac{\frac{1}{p_i}-a}{\frac{1}{p_i^*}-a}+\frac{1}{p_i-p_j^*}\frac{p_i^n}{p_j^{*n}}e^{\xi_i'+\xi_j'^*}\Big|_{N\times N},\end{aligned} \tag{36}$$

with $\xi_i'=i(\frac{p_i}{be^\theta}+\frac{e^\theta}{ap_i})t+\xi_{i,0}'$, and the parameters need to satisfy the constraint condition

$$1-ap_i=-\frac{p_i}{p_i^*}\Big(1-\frac{b}{p_i}\Big). \tag{37}$$

For example, taking $N=1$, the solution for the PT-symmetry IDNLS Eq. (1) reads

$$\psi_n=\sqrt{\frac{ab-1}{\delta ab}}e^{(n\theta-i\frac{e^\theta+e^{-\theta}}{ab}t+2it)}\cdot\frac{\frac{p_1^*(1-ap_1)}{p_1(1-ap_1^*)}+(\frac{p_1}{p_1^*})^ne^{\xi_1'+\xi_1'^*}}{1+(\frac{p_1}{p_1^*})^ne^{\xi_1'+\xi_1'^*}}, \tag{38}$$

with $\delta=1(ab<0)$ and $\delta=-1(0<ab<1)$. Here $\xi_i'=i(\frac{p_i}{be^\theta}+\frac{e^\theta}{ap_i})t+\xi_{i,0}'$, p_1 and $\xi_{1,0}'$ are arbitrary complex parameters, a, b and θ are arbitrary real parameters and it is necessary for them to satisfy $1-ap_1=-\frac{p_1}{p_1^*}(1-\frac{b}{p_1})$ and $ab<1$.

If we set $p_i = \exp(\alpha_i + i\beta_i)$, $[\frac{(1-ap_1^*)}{(1-ap_1)}] = e^{2i\gamma_1}$ and define $\xi_1 + \xi_1^* \equiv 2\vartheta_1$, then the modular square of ψ_n is given by

$$|\psi_n|^2 = \exp(2n\theta)\Big|\frac{1-ab}{ab}\Big| \cdot \Big[1 - \frac{2\sin(2n\beta_1+\beta_1+\gamma)\sin(\beta_1+\gamma)}{\cosh(2\vartheta_1)+\cos(2n\beta_1)}\Big], \tag{39}$$

which implies that the singularity of $|\psi_n|^2$ occurs at $\beta_1 = k\pi$ $(k = \pm1, \pm2, \ldots)$.

(C) The reverse time discrete symmetric IDNLS Eq. (1): For the nonlocal IDNLS Eq. (1), we fail to construct its general Casorati determinant solution. Here we list a one-soliton solution to show the reduction process. To be specific, taking $N = 1$, one has

$$u_n = \frac{\frac{q_1}{1-aq_1} + \frac{p_1}{1-ap_1}(\frac{p_1}{q_1})^n \frac{e^{\xi_1'}}{e^{\eta_1'}}}{1 + (\frac{p_1}{q_1})^n \frac{e^{\xi_1'}}{e^{\eta_1'}}},$$
$$v_n = \frac{\frac{1-ap_1}{p_1} + \frac{1-aq_1}{q_1}(\frac{q_1}{p_1})^n \frac{e^{\eta_1'}}{e^{\xi_1'}}}{1 + (\frac{q_1}{p_1})^n \frac{e^{\eta_1'}}{e^{\xi_1'}}}, \tag{40}$$

$$u_n(-t) = \frac{\frac{q_1}{1-aq_1} + \frac{p_1}{1-ap_1}(\frac{p_1}{q_1})^n \frac{e^{\eta_1'}}{e^{\xi_1'}}}{1 + (\frac{p_1}{q_1})^n \frac{e^{\eta_1'}}{e^{\xi_1'}}},$$
$$v_n(-t) = \frac{\frac{1-ap_1}{p_1} + \frac{1-aq_1}{q_1}(\frac{q_1}{p_1})^n \frac{e^{\xi_1'}}{e^{\eta_1'}}}{1 + (\frac{q_1}{p_1})^n \frac{e^{\xi_1'}}{e^{\eta_1'}}}, \tag{41}$$

with $\xi_i' = i(acp_i + \frac{bd}{p_i})t$ and $\eta_i' = i(acq_i + \frac{bd}{q_i})t$. To keep the term $(\frac{q_1}{p_1})^n$ in v_n as the same as the term $(\frac{p_1}{q_1})^n$ in $u_n(-t)$, only $q_1 = -p_1$ can be chosen, which yields $v_n = \frac{a^2p_1^2-1}{p_1^2}u_n(-t)$ and $u_n = \frac{p_1^2}{a^2p_1^2-1}v_n(-t)$. In this situation, the constraint condition (13) reduces to $p_1^2 = \frac{b}{a}$. Through the variable transformations

$$u_n = \frac{\psi_n}{\alpha}\exp\Big[2i(\frac{1}{ab}-1)t\Big], \quad \alpha^2 = \frac{(ab-1)^2}{\gamma b},$$
$$c = d = \frac{1}{ab}, \tag{42}$$

where a, b and α are complex constants, we obtain the reverse time discrete symmetric IDNLS Eq. (1). Thus a one-soliton solution has the following form

$$\psi_n = \frac{\alpha p_1}{1-ap_1}e^{-2i(\frac{1}{ab}-1)t}\frac{\frac{ap_1-1}{ap_1+1} + (-1)^n e^{\frac{2i}{ap_1}t}}{1+(-1)^n e^{\frac{2i}{ap_1}t}}, \tag{43}$$

with $\alpha^2 = \frac{(ab-1)^2}{\gamma b}$ and $p_1^2 = \frac{b}{a}$.

Note that for fixed a and b, the constraint condition $p_i^2 = \frac{b}{a}$ does not result in different values of the wave number p_i except for its opposite value. Therefore, in the manner of this reduction, we cannot obtain the general Casorati determinant solution of the nonlocal IDNLS Eq. (1). However, if we consider a pair reduction for the wave numbers p_i and q_i, the solution with even number $(2N)$ solitons can be derived. The details have been discussed in a separate work.

(D) The reverse discrete-time symmetric IDNLS Eq. (1): For the nonlocal IDNLS Eq. (1), we define the diagonal matrices and another Vandermonde matrix as follows:

$$A = \mathrm{Diag}(a_1, a_2\cdots, a_N),$$
$$B = \mathrm{Diag}(b_1, b_2\cdots, b_N),$$
$$B' = \mathrm{Diag}(b_1', b_2'\cdots, b_N'), \tag{44}$$
$$C = \mathrm{Diag}(c_1, c_2\cdots, c_N),$$
$$C' = \mathrm{Diag}(c_1', c_2'\cdots, c_N'),$$
$$D = \mathrm{Diag}(d_1, d_2\cdots, d_N), \tag{45}$$
$$D' = \mathrm{Diag}(d_1', d_2'\cdots, d_N'),$$

$$V_p = [(-1)^{N-i}\sum_{\substack{1\le k_1<k_2<\ldots<k_{N-i}\le N\\ k_l\neq j}}(\prod_{l=1}^{N-i}p_{k_l})]_{N\times N}, \tag{46}$$

with the elements $a_i = \frac{1}{p_i-q_i}$, $b_i = \frac{1}{q_i^n e^{\eta_i}}$, $b_i' = \frac{1}{p_i^n e^{\xi_i}}$, $c_i = \frac{(-1)^{N+1}}{\prod_{k\neq i}(p_k-q_i)}$, $c_i' = \frac{(-1)^{N+1}}{\prod_{k\neq i}(q_k-p_i)}$, $d_i = \frac{p_i^{1/2}}{(1-ap_i)^{1/2}}$ and $d_i' = \frac{q_i^{1/2}}{(1-aq_i)^{1/2}}$. In this case, the exponential functions in Casorati determinants are changed as

$$\exp(\xi_i) = \prod_{\substack{k=1\\k\neq i}}^{N}(p_k-q_i)^{1/2}(p_k-p_i)^{-1/2}\exp(\xi_i'),$$
$$\exp(\eta_i) = \prod_{\substack{k=1\\k\neq i}}^{N}(q_k-p_i)^{1/2}(q_k-q_i)^{-1/2}\exp(\eta_i'), \tag{47}$$

with $\xi_i' = i(acp_i + \frac{bd}{p_i})t$ and $\eta_i' = i(acq_i + \frac{bd}{q_i})t$.

Based on the gauge freedom of tau functions, it can be checked that

$$\tilde f_n = |\tilde F| = |CABFV_pA^{-1}| = \Big|1+\Gamma_i\frac{p_i^n e^{\xi_i'}}{q_i^n e^{\eta_i'}}\ (i=j) \text{ or } \frac{p_j-q_j}{p_j-q_i}\ (i\neq j)\Big|_{N\times N}, \tag{48}$$

$$\tilde g_n = |\tilde G| = |D'^{-1}CABGV_pA^{-1}D^{-1}| = \Big|\frac{Q_i}{P_i}+\frac{P_i}{Q_i}\Gamma_i\frac{p_i^n e^{\xi_i'}}{q_i^n e^{\eta_i'}}\ (i=j) \text{ or } \frac{Q_i}{P_j}\frac{p_j-q_j}{p_j-q_i}\ (i\neq j)\Big|_{N\times N}, \tag{49}$$

$$\tilde h_n = |\tilde H| = |D'CABHV_pA^{-1}D| = \Big|\frac{P_i}{Q_i}+\frac{Q_i}{P_i}\Gamma_i\frac{p_i^n e^{\xi_i'}}{q_i^n e^{\eta_i'}}\ (i=j) \text{ or } \frac{P_j}{Q_i}\frac{p_j-q_j}{p_j-q_i}\ (i\neq j)\Big|_{N\times N}, \tag{50}$$

with $P_i = \frac{p_i^{1/2}}{(1-ap_i)^{1/2}}$, $Q_i = \frac{q_i^{1/2}}{(1-aq_i)^{1/2}}$ and $\Gamma_i = \prod_{k=1,k\neq i}^{N}(p_k-p_i)^{1/2}(q_k-q_i)^{1/2}(p_k-q_i)^{-1/2}(q_k-p_i)^{-1/2}$, still satisfying the bilinear Eqs. (7)–(9).

On the other hand, using another kind of transformation, we can obtain

$$\tilde{f}'_n = |F'| = \Big|C'B'FV_q\Big|$$
$$= \Big|1 + \varGamma_i \frac{q_i^n e^{\eta_i'}}{p_i^n e^{\xi_i'}}\,(i=j) \text{ or } \frac{p_i - q_i}{p_i - q_j}\,(i \neq j)\Big|_{N\times N}, \tag{51}$$
$$\tilde{h}'_n = |H'| = |DC'B'HV_qD'|$$
$$= \Big|\frac{Q_i}{P_i} + \frac{P_i}{Q_i}\varGamma_i \frac{q_i^n e^{\eta_i'}}{p_i^n e^{\xi_i'}}\,(i=j) \text{ or } \frac{Q_j}{P_i}\frac{p_i - q_i}{p_i - q_j}\,(i\neq j)\Big|_{N\times N}. \tag{52}$$

Then, it can be found that

$$F_{ij}(-n,-t) = F'_{ji}, \quad G_{ij}(-n,-t) = H'_{ji}, \tag{53}$$

which yields

$$\tilde{f}_{-n}(-t) = \tilde{f}'_n = |C'B'(B^{-1}A^{-1}C^{-1}\tilde{F}AV_p^{-1})V_q| = \frac{|B'||C'||V_q|}{|B||C||V_p|}\tilde{f}_n, \tag{54}$$
$$\tilde{g}_{-n}(-t) = \tilde{h}'_n = |DC'B'(B^{-1}A^{-1}C^{-1}D^{'-1} \cdot \tilde{H}D^{-1}AV_p^{-1})V_qD^{'}| = \frac{|B'||C'||V_q|}{|B||C||V_p|}\tilde{h}_n. \tag{55}$$

Thus it immediately reaches $u_{-n}(-t) = \frac{\tilde{g}_{-n}(-t)}{\tilde{f}_{-n}(-t)} = \frac{\tilde{h}_n}{\tilde{f}_n} = v_n$.

Furthermore, through the variable transformations

$$u_n = \frac{\psi_n}{\alpha}\exp\Big(-n\theta + i\frac{e^{\theta}+e^{-\theta}}{ab}t - 2it\Big), \quad \alpha^2 = \frac{ab-1}{\gamma ab}, \quad c = \frac{e^{-\theta}}{ab}, \quad d = \frac{e^{\theta}}{ab}, \tag{56}$$

where a, b, α and θ are complex constants, we obtain the reverse discrete-time symmetric IDNLS Eq. (1). Finally, we obtain the following theorem regarding the solution to the nonlocal IDNLS Eq. (1).

Theorem 1.3: The reverse discrete-time symmetric IDNLS Eq. (1) has the following solution

$$\psi_n = \alpha e^{(n\theta - i\frac{e^{\theta}+e^{-\theta}}{ab}t + 2it)}\frac{\tilde{g}_n}{\tilde{f}_n}, \quad \alpha^2 = \frac{ab-1}{\gamma ab}, \tag{57}$$

where

$$\tilde{f}_n = \Big|1 + \varGamma_i \frac{p_i^n e^{\xi_i'}}{q_i^n e^{\eta_i'}}\,(i=j) \quad \text{or} \quad \frac{p_j - q_j}{p_j - q_i}\,(i\neq j)\Big|_{N\times N},$$
$$\tilde{g}_n = \Big|\frac{Q_i}{P_i} + \frac{P_i}{Q_i}\varGamma_i \frac{p_i^n e^{\xi_i'}}{q_i^n e^{\eta_i'}}\,(i=j) \text{ or } \frac{Q_i}{P_j}\frac{p_j - q_j}{p_j - q_i}\,(i\neq j)\Big|_{N\times N}, \tag{58}$$

with $\xi_i' = i(\frac{p_i}{be^{\theta}} + \frac{e^{\theta}}{ap_i})t$, $\eta_i' = i(\frac{q_i}{be^{\theta}} + \frac{e^{\theta}}{aq_i})t$, $P_i = \frac{p_i^{1/2}}{(1-ap_i)^{1/2}}$, $Q_i = \frac{q_i^{1/2}}{(1-aq_i)^{1/2}}$ and $\varGamma_i = \prod_{k=1,k\neq i}^{N}(p_k - p_i)^{1/2}(q_k - q_i)^{1/2}(p_k - q_i)^{-1/2}(q_k - p_i)^{-1/2}$. Here a, b, θ, p_i and q_i are arbitrary parameters, and they need to satisfy the constraint condition (13),

$$q_i = \frac{b - p_i}{1 - ap_i}. \tag{59}$$

For example, taking $N = 1$, the solution to reverse discrete-time symmetric IDNLS Eq. (1) is given by

$$\psi_n = \alpha e^{(n\theta - i\frac{e^{\theta}+e^{-\theta}}{ab}t + 2it)}\frac{\frac{Q_1}{P_1} + \frac{P_1}{Q_1}(\frac{p_1}{q_1})^n e^{\xi_1' - \eta_1'}}{1 + (\frac{p_1}{q_1})^n e^{\xi_1' - \eta_1'}}, \tag{60}$$

with $\alpha^2 = \frac{ab-1}{\gamma ab}$, $P_1 = \frac{p_1^{1/2}}{(1-ap_1)^{1/2}}$, $Q_1 = \frac{q_1^{1/2}}{(1-aq_1)^{1/2}}$, $\xi_1' = i(\frac{p_1}{be^{\theta}} + \frac{e^{\theta}}{ap_1})t$ and $\eta_1' = i(\frac{q_1}{be^{\theta}} + \frac{e^{\theta}}{aq_1})t$, where a, b, θ, p_1 and q_1 are arbitrary complex parameters, and they need to satisfy the constraint condition $q_1 = \frac{b-p_1}{1-ap_1}$.

In summary, we have investigated solutions to local and nonlocal IDNLS equations via reduction. Due to the gauge invariance for the before-reduction equations, tau functions in terms of the single Casorati determinant are written as several alternative forms. Based on the different forms of the Casorati determinant, solutions to local and nonlocal IDNLS equations are derived by imposing the corresponding constraint conditions.

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