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Multi-Soliton Solutions for the Coupled Fokas–Lenells System via Riemann–Hilbert Approach *

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We aim to construct multi-soliton solutions for the coupled Fokas–Lenells system which arises as a model for describing the nonlinear pulse propagation in optical fibers. Starting from the spectral analysis of the Lax pair, a Riemann–Hilbert problem is presented. Then in the framework of the Riemann–Hilbert problem corresponding to the reflectionless case, N-soliton solutions to the coupled Fokas–Lenells system are derived explicitly.

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The so-called coupled Fokas–Lenells (FL) system[1] reads

$$\begin{pmatrix} p_1 \\ p_2 \\ r_1 \\ r_2 \end{pmatrix}_t = i\begin{pmatrix} \gamma u_{1,xx} - 2p_1u_1v_1 - p_1u_2v_2 - p_2u_1v_2 \\ \gamma u_{2,xx} - 2p_2u_2v_2 - p_2u_1v_1 - p_1u_2v_1 \\ -\gamma v_{1,xx} + 2r_1u_1v_1 + r_1u_2v_2 + r_2u_2v_2 \\ -\gamma v_{2,xx} + 2r_2u_2v_2 + r_2u_1v_1 + r_1u_1v_2 \end{pmatrix}, \tag{1}$$

with

$$p_k = u_k + i\upsilon u_{k,x},\ \ r_k = v_k - i\upsilon v_{k,x},\ \ k = 1, 2,$$

where u_k and v_k are complex-valued functions of x, t, while γ and υ are nonzero real parameters. The coupled FL system (1) is an integrable generalization of Manakov's system that is characterized by an equal nonlinear interaction between two components. Upon employing the transformations $v_k = -u_k^*$, $u_k = e^{ix}q_k$, $q_1 = q$, $q_2 = r$ as well as $\gamma = 2$, $\upsilon = 1$, the coupled FL system (1) is then turned into the following form

$$iq_{xt} - 2iq_{xx} + 4q_x - (2|q|^2 + |r|^2)q_x - qr^*r_x + 2iq = 0, \tag{2a}$$

$$ir_{xt} - 2ir_{xx} + 4r_x - (2|r|^2 + |q|^2)r_x - rq^*q_x + 2ir = 0, \tag{2b}$$

which describes the nonlinear pulse propagation in optical fibers by retaining terms up to the next leading asymptotic order. Here the asterisk represents complex conjugation. Much of the research has been carried out on the coupled FL system (2). For example, Zhang *et al.*[2] performed the n-fold Darboux transformation method to derive diverse solutions, including the higher-order soliton, breather and rogue wave solutions. In a follow-up study conducted by Ling *et al.*,[3] it was shown that the coupled FL system (2) possesses a multi-Hamiltonian structure and infinitely many conservation laws. Based on a generalized Darboux transformation and a limiting process, different kinds of one-soliton solutions were revealed, some of which include bright-dark solitons, dark-anti-dark solitons as well as breather-like solutions. In addition, the multi-dark soliton solutions to the coupled FL system (2) were found by applying the limit technique.

Of particular concern in the field of nonlinear science is to find multi-soliton solutions for nonlinear partial differential equations (NLPDEs). So far, a number of approaches are available for achieving the goal, including the inverse scattering method,[4,5] the Hirota bilinear method,[6–8] the Darboux transformation method,[9–11] the Riemann–Hilbert approach[12–19] and so forth. In recent years, researchers have shown an increasing interest in applying the Riemann–Hilbert approach to treat NLPDEs under initial-boundary value conditions. For example, the coupled derivative Schrödinger equation was investigated in Ref. [20] and a compact N-soliton solution formula was found. Based on this result, some properties of the one-soliton solution and asymptotic analysis of the N-soliton solution were discussed. More recently, Zhang *et al.*[21] examined the complex Sharma–Tasso–Olver equation on half line and showed that the solution to the equation can be expressed in terms of the solution of a Riemann–Hilbert problem. The objective of this research is to explore multi-soliton solutions for the coupled FL system (2) by use of the Riemann–Hilbert approach.

This section seeks to set up a Riemann–Hilbert problem for the coupled FL system (2). We begin our discussion by considering the Lax pair representation[2] of the coupled FL system (2),

$$\varPhi_x = U\varPhi = (i\varsigma^2\sigma + \varsigma Q)\varPhi, \tag{3a}$$

$$\varPhi_t = V\varPhi = \Big(2i\varsigma^2\sigma + 2\varsigma Q - 2i\sigma + iV_0 + iV_{-1}\varsigma^{-1} + \frac{1}{2}i\varsigma^{-2}\sigma\Big)\varPhi, \tag{3b}$$

where $\varPhi = (\psi(x,t), \phi(x,t), \varphi(x,t))^{\rm T}$ is the vector eigenfunction, the symbol T denotes transpose of the

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vector, and $\varsigma \in \mathbb{C}$ is a spectral parameter. Furthermore, $\sigma = \mathrm{diag}(-1, 1, 1)$,

$$Q = \begin{pmatrix} 0 & q_x & r_x \\ q_x^* & 0 & 0 \\ r_x^* & 0 & 0 \end{pmatrix},$$
$$V_0 = \begin{pmatrix} -|q|^2 - |r|^2 & 0 & 0 \\ 0 & |q|^2 & q^* r \\ 0 & q r^* & |r|^2 \end{pmatrix},$$
$$V_{-1} = \begin{pmatrix} 0 & q & r \\ -q^* & 0 & 0 \\ -r^* & 0 & 0 \end{pmatrix}.$$

The Lax pair (3) can be further written as the equivalent form

$$\varPhi_x = (i\varsigma^2\sigma + U_1)\varPhi, \tag{4a}$$
$$\varPhi_t = (2i\kappa^2\sigma + U_2)\varPhi, \tag{4b}$$

where

$$\kappa = \varsigma - \frac{1}{2\varsigma},\ U_1 = \varsigma Q,$$
$$U_2 = 2\varsigma Q + iV_0 + iV_{-1}\varsigma^{-1}.$$

For our analysis, we here extend $\varPhi$ into a matrix and introduce the variable transformation

$$\varPhi = J e^{i\varsigma^2\sigma x + 2i\kappa^2\sigma t}, \tag{5}$$

with $J = J(x, t; \varsigma)$ being a new matrix spectral function. Using Eq. (5), the Lax pair (4) we shall deal with is written as

$$J_x = i\varsigma^2[\sigma, J] + U_1 J, \tag{6a}$$
$$J_t = 2i\kappa^2[\sigma, J] + U_2 J, \tag{6b}$$

where $[\sigma, J] = \sigma J - J\sigma$.

As to the direct scattering process, we will restrict our attention to the spectral problem (6a), and the variable t enters as a dummy variable. Now, the matrix Jost solutions $J_\pm$ for Eq. (6a) are introduced as a collection of columns, that is,

$$J_\pm = ([J_\pm]_1, [J_\pm]_2, [J_\pm]_3), \tag{7}$$

where $J_\pm$ are uniquely determined by the integral equations of the Volterra type

$$J_- = \mathbb{I} + \int_{-\infty}^{x} e^{i\varsigma^2\sigma(x-\xi)} U_1(\xi) J_-(\xi, \varsigma) \cdot e^{-i\varsigma^2\sigma(x-\xi)} d\xi, \tag{8a}$$
$$J_+ = \mathbb{I} - \int_{x}^{+\infty} e^{i\varsigma^2\sigma(x-\xi)} U_1(\xi) J_+(\xi, \varsigma) \cdot e^{-i\varsigma^2\sigma(x-\xi)} d\xi, \tag{8b}$$

obeying the asymptotic conditions

$$J_- \to \mathbb{I},\ x \to -\infty, \tag{9a}$$
$$J_+ \to \mathbb{I},\ x \to +\infty, \tag{9b}$$

and $\mathbb{I}$ denotes the 3×3 identity matrix. Analyzing Eq. (8) directly reveals that $[J_+]_1, [J_-]_2$ and $[J_-]_3$ are analytic for $\varsigma \in \mathbb{D}^-$ and continuous for $\varsigma \in \mathbb{D}^- \cup \mathbb{R} \cup i\mathbb{R}$, whereas $[J_-]_1, [J_+]_2$ and $[J_+]_3$ are analytic for $\varsigma \in \mathbb{D}^+$ and continuous for $\varsigma \in \mathbb{D}^+ \cup \mathbb{R} \cup i\mathbb{R}$, where

$$\mathbb{D}^+ = \Big\{\varsigma \big| \arg\varsigma \in \Big(0, \frac{\pi}{2}\Big) \cup \Big(\pi, \frac{3\pi}{2}\Big)\Big\},$$
$$\mathbb{D}^- = \Big\{\varsigma \big| \arg\varsigma \in \Big(\frac{\pi}{2}, \pi\Big) \cup \Big(\frac{3\pi}{2}, 2\pi\Big)\Big\}.$$

Indeed, the determinants of $J_\pm$ are independent for all x in light of Abel's identity and $\mathrm{tr}Q = 0$. Based on the asymptotic conditions (9), we thus have $\det J_\pm = 1$ for $\varsigma \in \mathbb{R} \cup i\mathbb{R}$.

Since both $J_- E$ and $J_+ E$ are matrix solutions of the spectral problem (4a) for $\varsigma \in \mathbb{R} \cup i\mathbb{R}$, where $E = e^{i\varsigma^2\sigma x}$, they must be linearly related by a 3×3 scattering matrix $S(\varsigma) = (s_{kj})_{3\times3}$, namely

$$J_- E = J_+ E \cdot S(\varsigma),\ \varsigma \in \mathbb{R} \cup i\mathbb{R}. \tag{10}$$

It is easy to know $\det S(\varsigma) = 1$. Furthermore, we can see from the analytic property of J_- that s_{11} admits an analytic extension to $\mathbb{D}^+$, and s_{kj} $(k, j = 2, 3)$ extend to $\mathbb{D}^-$ analytically.

A Riemann–Hilbert problem to be formulated for the coupled FL system (2) requires two matrix functions: one is analytic in $\mathbb{D}^+$ and the other is analytic in $\mathbb{D}^-$. Let $P_1 = P_1(x, \varsigma)$ be an analytic function of ς,

$$P_1(x, \varsigma) = ([J_-]_1, [J_+]_2, [J_+]_3)(x, \varsigma), \tag{11}$$

defining in $\mathbb{D}^+$. Then, we can investigate the very large-ς asymptotic behavior of P_1 having the asymptotic expansion

$$P_1 = P_1^{(0)} + \frac{P_1^{(1)}}{\varsigma} + \frac{P_1^{(2)}}{\varsigma^2} + O\Big(\frac{1}{\varsigma^3}\Big),\quad \varsigma \to \infty. \tag{12}$$

Inserting Eq. (12) into Eq. (6a) and equating terms with like powers of ς, we have

$$O(1): i[\sigma, P_1^{(2)}] + Q P_1^{(1)} = P_{1x}^{(0)},$$
$$O(\varsigma): i[\sigma, P_1^{(1)}] + Q P_1^{(0)} = 0,$$
$$O(\varsigma^2): i[\sigma, P_1^{(0)}] = 0,$$

from which we find $P_1^{(0)} = \mathbb{I}$, namely, $P_1 \to \mathbb{I}$ as $\varsigma \in \mathbb{D}^+ \to \infty$.

To formulate a Riemann–Hilbert problem for the coupled FL system (2), we will manage to find the analytic counterpart of P_1 in $\mathbb{D}^-$. The so-called adjoint scattering equation of (6a) takes the form

$$K_x = i\varsigma^2[\sigma, K] - K U_1. \tag{13}$$

Here we write the inverse matrices of $J_\pm$ as a collection of rows,

$$J_\pm^{-1} = \begin{pmatrix} [J_\pm^{-1}]^1 \\ [J_\pm^{-1}]^2 \\ [J_\pm^{-1}]^3 \end{pmatrix}. \tag{14}$$

It can be seen that $J_{\pm}^{-1}$ fulfill Eq. (13) and obey the boundary conditions $J_{\pm}^{-1} \to \mathbb{I}$ as $x \to \pm\infty$. From Eq. (10), it follows

$$E^{-1}J_{-}^{-1} = R(\varsigma)\cdot E^{-1}J_{+}^{-1}, \tag{15}$$

where $R(\varsigma) = (r_{kj})_{3\times 3} = S^{-1}(\varsigma)$. The matrix function P_2, which is analytic in $\mathbb{D}^-$, can be defined as

$$P_2(x,\varsigma) = \begin{pmatrix} [J_{-}^{-1}]^1 \\ [J_{+}^{-1}]^2 \\ [J_{+}^{-1}]^3 \end{pmatrix}(x,\varsigma). \tag{16}$$

Similar to P_1, we can also obtain the very large-ς asymptotic behavior $P_2 \to \mathbb{I}$ as $\varsigma \in \mathbb{D}^- \to \infty$. Moreover, it can be shown that r_{11} accepts an analytic extension to $\mathbb{D}^-$ and r_{kj} $(k,j = 2,3)$ extend to $\mathbb{D}^+$ analytically.

Insertion of Eq. (7) into Eq. (10) gives

$$\begin{aligned}&([J_-]_1,[J_-]_2,[J_-]_3)\\ &=([J_+]_1,[J_+]_2,[J_+]_3)\\ &\quad\cdot\begin{pmatrix} s_{11} & s_{12}e^{-2i\varsigma^2 x} & s_{13}e^{-2i\varsigma^2 x} \\ s_{21}e^{2i\varsigma^2 x} & s_{22} & s_{23} \\ s_{31}e^{2i\varsigma^2 x} & s_{32} & s_{33}\end{pmatrix},\end{aligned}$$

from which the expression of $[J_-]_1$ reads

$$[J_-]_1 = s_{11}[J_+]_1 + s_{21}e^{2i\varsigma^2 x}[J_+]_2 + s_{31}e^{2i\varsigma^2 x}[J_+]_3.$$

Thus P_1 is of the form

$$P_1 = ([J_-]_1,[J_+]_2,[J_+]_3) = J_+\begin{pmatrix} s_{11} & 0 & 0 \\ s_{21}e^{2i\varsigma^2 x} & 1 & 0 \\ s_{31}e^{2i\varsigma^2 x} & 0 & 1\end{pmatrix}.$$

On the other hand, putting Eq. (14) into Eq. (15), we derive

$$\begin{aligned}\begin{pmatrix} [J_{-}^{-1}]^1 \\ [J_{-}^{-1}]^2 \\ [J_{-}^{-1}]^3 \end{pmatrix} &= \begin{pmatrix} r_{11} & r_{12}e^{-2i\varsigma^2 x} & r_{13}e^{-2i\varsigma^2 x} \\ r_{21}e^{2i\varsigma^2 x} & r_{22} & r_{23} \\ r_{31}e^{2i\varsigma^2 x} & r_{32} & r_{33}\end{pmatrix}\\ &\quad\cdot\begin{pmatrix} [J_{+}^{-1}]^1 \\ [J_{+}^{-1}]^2 \\ [J_{+}^{-1}]^3 \end{pmatrix},\end{aligned}$$

from which we can express $[J_{-}^{-1}]^1$ as

$$\begin{aligned}[J_{-}^{-1}]^1 =& r_{11}[J_{+}^{-1}]^1 + r_{12}e^{-2i\varsigma^2 x}[J_{+}^{-1}]^2\\ &+ r_{13}e^{-2i\varsigma^2 x}[J_{+}^{-1}]^3.\end{aligned}$$

Consequently, P_2 takes the form

$$\begin{aligned}P_2 &= \begin{pmatrix} [J_{-}^{-1}]^1 \\ [J_{+}^{-1}]^2 \\ [J_{+}^{-1}]^3 \end{pmatrix}\\ &= \begin{pmatrix} r_{11} & r_{12}e^{-2i\varsigma^2 x} & r_{13}e^{-2i\varsigma^2 x} \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix} J_{+}^{-1}.\end{aligned}$$

By now, we have found two matrix functions P_1 and P_2, which are analytic in $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively. What follows is a description of the Riemann–Hilbert problem for the coupled FL system (2). By denoting that the limit of P_1 is P^+ when $\varsigma \in \mathbb{D}^+$ approaches $\mathbb{R}\cup i\mathbb{R}$ and the limit of P_2 is P^- when $\varsigma \in \mathbb{D}^-$ approaches $\mathbb{R}\cup i\mathbb{R}$, the Riemann–Hilbert problem we are looking for can be set up as follows:

$$\begin{aligned}&P^-(x,\varsigma)P^+(x,\varsigma)\\ &= \begin{pmatrix} 1 & r_{12}e^{-2i\varsigma^2 x} & r_{13}e^{-2i\varsigma^2 x} \\ s_{21}e^{2i\varsigma^2 x} & 1 & 0 \\ s_{31}e^{2i\varsigma^2 x} & 0 & 1\end{pmatrix},\end{aligned} \tag{17}$$

under canonical normalization conditions

$$\begin{aligned}&P_1(x,\varsigma)\to\mathbb{I},\quad \varsigma\in\mathbb{D}^+\to\infty,\\ &P_2(x,\varsigma)\to\mathbb{I},\quad \varsigma\in\mathbb{D}^-\to\infty,\end{aligned}$$

and $r_{11}s_{11} + r_{12}s_{21} + r_{13}s_{31} = 1$.

What we set out to carry out now is to solve the Riemann–Hilbert problem (17) that is assumed to be irregular. The irregularity indicates that both $\det P_1$ and $\det P_2$ have certain zeros in their analytic domains. Resorting to the definitions of P_1 and P_2 as well as the scattering relation (10) yields

$$\begin{aligned}&\det P_1(\varsigma) = s_{11}(\varsigma),\quad \varsigma\in\mathbb{D}^+,\\ &\det P_2(\varsigma) = r_{11}(\varsigma),\quad \varsigma\in\mathbb{D}^-,\end{aligned}$$

which enable us to know that the zeros of $\det P_1$ and $\det P_2$ are the same as s_{11} and r_{11}, respectively.

Based on the above facts, it is necessary to analyze the characteristics of the zeros. With regard to the potential matrix Q, there exists the symmetry relation $Q^\dagger = -\sigma Q\sigma$, where the superscript $\dagger$ stands for the Hermitian of a matrix, and $\sigma = \mathrm{diag}(-1,1,1)$. It will be convenient for our analysis to rewrite Eqs. (11) and (16) as

$$P_1 = J_-H_1 + J_+H_2, \tag{18a}$$
$$P_2 = H_1J_{-}^{-1} + H_2J_{+}^{-1}, \tag{18b}$$

where $H_1 = \mathrm{diag}(1,0,0)$ and $H_2 = \mathrm{diag}(0,1,1)$. In view of $J_{\pm}^{\dagger} = \sigma J_{\pm}^{-1}\sigma$, it can be seen from Eq. (18) that

$$P_1^\dagger = \sigma P_2\sigma, \tag{19}$$

and the involution property of the scattering matrix $S^\dagger(\varsigma^*) = \sigma S^{-1}(\varsigma)\sigma$, which further gives

$$s_{11}^*(\varsigma^*) = r_{11}(\varsigma),\quad \varsigma\in\mathbb{D}^-. \tag{20}$$

This illustrates that each zero $\pm\varsigma_k$ of s_{11} results in each zero $\pm\varsigma_k^*$ of r_{11} correspondingly.

Additionally, the potential matrix Q meets the relation $Q = -\sigma Q\sigma$, based on which we can conclude $J_\pm(\varsigma) = \sigma J_\pm(-\varsigma)\sigma$ and

$$P_1(\varsigma) = \sigma P_1(-\varsigma)\sigma. \tag{21}$$

Also we have $S(\varsigma)=\sigma S(-\varsigma)\sigma$, which leads to $s_{11}(\varsigma)=s_{11}(-\varsigma)$ for $\varsigma\in\mathbb{D}^+$, namely $s_{11}(\pm\varsigma_k)=0$. At this point, we posit that $\det P_1$ has $2N$ simple zeros $\{\varsigma_j\}$ $(1\le j\le 2N)$ in $\mathbb{D}^+$, where

$$\varsigma_{N+l}=-\varsigma_l,\quad 1\le l\le N, \tag{22}$$

and $\det P_2$ has $2N$ simple zeros $\{\hat\varsigma_j\}$ $(1\le j\le 2N)$ in $\mathbb{D}^-$, where

$$\hat\varsigma_l=\varsigma_l^*,\quad 1\le l\le 2N. \tag{23}$$

Actually, the scattering data needed to solve the Riemann–Hilbert problem (17) include the continuous scattering data $\{s_{21},s_{31}\}$ as well as the discrete scattering data $\{\varsigma_j,\hat\varsigma_j,\nu_j,\hat\nu_j\}$, in which ν_j and $\hat\nu_j$ are nonzero column vectors and row vectors, respectively, meeting

$$P_1(\varsigma_j)\nu_j=0, \tag{24a}$$
$$\hat\nu_j P_2(\hat\varsigma_j)=0. \tag{24b}$$

Performing the Hermitian of Eq. (24a) and taking into account the involution properties of Eqs. (19) and (23), we can reveal the relation

$$\hat\nu_j=\nu_j^\dagger\sigma,\quad 1\le j\le 2N.$$

According to Eqs. (21), (22) and (24a), we have

$$\nu_j=\sigma\nu_{j-N},\quad N+1\le j\le 2N.$$

For the purpose of obtaining the explicit form of the vectors ν_j, we calculate the derivatives of Eq. (24a) with respect to x and t and find

$$\frac{\partial\nu_j}{\partial x}=i\varsigma_j^2\sigma\nu_j,$$
$$\frac{\partial\nu_j}{\partial t}=2i\kappa^2\sigma\nu_j.$$

Thus the vectors ν_j and $\hat\nu_j$ are derived as

$$\nu_j=\begin{cases} e^{[i\varsigma_j^2x+2i(\varsigma_j-\frac{1}{2\varsigma_j})^2t]\sigma}\nu_{j,0}, & 1\le j\le N,\\ \sigma e^{[i\varsigma_{j-N}^2x+2i(\varsigma_{j-N}-\frac{1}{2\varsigma_{j-N}})^2t]\sigma}\nu_{j-N,0}, & N+1\le j\le 2N,\end{cases}$$

and

$$\hat\nu_j=\begin{cases} \nu_{j,0}^\dagger e^{[i\varsigma_j^2x+2i(\varsigma_j-\frac{1}{2\varsigma_j})^2t]^*\sigma}\sigma, & 1\le j\le N,\\ \nu_{j-N,0}^\dagger e^{[i\varsigma_{j-N}^2x+2i(\varsigma_{j-N}-\frac{1}{2\varsigma_{j-N}})^2t]^*\sigma}, & N+1\le j\le 2N,\end{cases}$$

where $\nu_{j,0}$ are the complex constant vectors.

It is pointed out that the Riemann–Hilbert problem (17) examined corresponds to the reflectionless case, namely, $s_{21}=s_{31}=0$. By introducing a $2N\times 2N$ matrix M with entries

$$m_{kj}=\frac{\hat\nu_k\nu_j}{\varsigma_j-\hat\varsigma_k},\quad 1\le k,\quad j\le 2N,$$

and assuming that the inverse matrix of M exists, then the solution for the Riemann–Hilbert problem (17) can be given by

$$P_1(\varsigma)=\mathbb{I}-\sum_{k=1}^{2N}\sum_{j=1}^{2N}\frac{\nu_k\hat\nu_j(M^{-1})_{kj}}{\varsigma-\hat\varsigma_j}, \tag{25a}$$
$$P_2(\varsigma)=\mathbb{I}+\sum_{k=1}^{2N}\sum_{j=1}^{2N}\frac{\nu_k\hat\nu_j(M^{-1})_{kj}}{\varsigma-\varsigma_k}. \tag{25b}$$

Furthermore, from Eq. (25a) we directly obtain

$$P_1^{(1)}=-\sum_{k=1}^{2N}\sum_{j=1}^{2N}\nu_k\hat\nu_j(M^{-1})_{kj}.$$

Next we are ready to construct the potential functions q and r with the aid of the scattering data. Substituting the asymptotic expansion

$$P_1(\varsigma)=\mathbb{I}+\frac{P_1^{(1)}}{\varsigma}+\frac{P_1^{(2)}}{\varsigma^2}+O\Big(\frac{1}{\varsigma^3}\Big),\ \varsigma\to\infty,$$

into Eq. (6a) gives

$$\begin{aligned}Q=&-i[\sigma,P_1^{(1)}]\\ =&\begin{pmatrix}0 & 2i(P_1^{(1)})_{12} & 2i(P_1^{(1)})_{13}\\ -2i(P_1^{(1)})_{21} & 0 & 0\\ -2i(P_1^{(1)})_{31} & 0 & 0\end{pmatrix},\end{aligned}$$

where $(P_1^{(1)})_{kj}$ denote the (k,j)-entry of matrix $P_1^{(1)}$. As a result, the expression of general N-soliton solutions for the coupled FL system (2) reads

$$\begin{aligned}q(x,t)=2i\int_x^\infty\Big[&\sum_{k=1}^{N}\sum_{j=1}^{N}\alpha_k\beta_j^*e^{-\xi_k+\xi_j^*}(M^{-1})_{kj}\\ &+\sum_{k=1}^{N}\sum_{j=N+1}^{2N}\alpha_k\beta_{j-N}^*e^{-\xi_k+\xi_{j-N}^*}(M^{-1})_{kj}\\ &-\sum_{k=N+1}^{2N}\sum_{j=1}^{N}\alpha_{k-N}\beta_j^*e^{-\xi_{k-N}+\xi_j^*}(M^{-1})_{kj}\\ &-\sum_{k=N+1}^{2N}\sum_{j=N+1}^{2N}\alpha_{k-N}\beta_{j-N}^*e^{-\xi_{k-N}+\xi_{j-N}^*}\\ &\cdot(M^{-1})_{kj}\Big]d\tilde x,\end{aligned}$$

$$
\begin{aligned}
r(x,t) = 2i\int_x^{\infty} \Big[& \sum_{k=1}^{N}\sum_{j=1}^{N} \alpha_k \gamma_j^* e^{-\xi_k+\xi_j^*} (M^{-1})_{kj} \\
& + \sum_{k=1}^{N}\sum_{j=N+1}^{2N} \alpha_k \gamma_{j-N}^* e^{-\xi_k+\xi_{j-N}^*} (M^{-1})_{kj} \\
& - \sum_{k=N+1}^{2N}\sum_{j=1}^{N} \alpha_{k-N} \gamma_j^* e^{-\xi_{k-N}+\xi_j^*} (M^{-1})_{kj} \\
& - \sum_{k=N+1}^{2N}\sum_{j=N+1}^{2N} \alpha_{k-N} \gamma_{j-N}^* e^{-\xi_{k-N}+\xi_{j-N}^*} \\
& \cdot (M^{-1})_{kj} \Big] d\tilde{x},
\end{aligned}
$$

with

$$
m_{kj} = \begin{cases}
[(\beta_k^* \beta_j + \gamma_k^* \gamma_j) e^{\xi_k^*+\xi_j} \\
-\alpha_k^* \alpha_j e^{-\xi_k^*-\xi_j}]/[\varsigma_j - \varsigma_k^*], \\
\qquad 1 \le k, j \le N, \\
[(\beta_k^* \beta_{j-N} + \gamma_k^* \gamma_{j-N}) e^{\xi_k^*+\xi_{j-N}} \\
+\alpha_k^* \alpha_{j-N} e^{-\xi_k^*-\xi_{j-N}}]/[-\varsigma_{j-N} - \varsigma_k^*], \\
\qquad 1 \le k \le N, N+1 \le j \le 2N, \\
[(\beta_{k-N}^* \beta_j + \gamma_{k-N}^* \gamma_j) e^{\xi_{k-N}^*+\xi_j} \\
+\alpha_{k-N}^* \alpha_j e^{-\xi_{k-N}^*-\xi_j}]/[\varsigma_j + \varsigma_{k-N}^*], \\
\qquad N+1 \le k \le 2N,\ 1 \le j \le N, \\
[(\beta_{k-N}^* \beta_{j-N} + \gamma_{k-N}^* \gamma_{j-N}) e^{\xi_{k-N}^*+\xi_{j-N}} \\
-\alpha_{k-N}^* \alpha_{j-N} e^{-\xi_{k-N}^*-\xi_{j-N}}]/[-\varsigma_{j-N} \\
+\varsigma_{k-N}^*], \qquad N+1 \le k,\ j \le 2N,
\end{cases}
$$

where we have chosen nonzero vectors $\nu_{k,0} = (\alpha_k, \beta_k, \gamma_k)^{\mathrm{T}}$, and $\xi_k = i\varsigma_k^2 x + 2i(\varsigma_k - \frac{1}{2\varsigma_k})^2 t$.

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