CHIN. PHYS. LETT. Vol. 34, No. 7 (2017) 070203

Fermionic Covariant Prolongation Structure for a Super Nonlinear Evolution Equation in 2+1 Dimensions *

Zhao-Wen Yan(颜昭雯)[1**], Xiao-Li Wang(王晓丽)[2], Min-Li Li(李民丽)[3]

[1] *School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021*
[2] *School of Science, Qilu University of Technology, Ji'nan 250353*
[3] *School of Mathematical Sciences, Capital Normal University, Beijing 100048*

(Received 14 April 2017)

The integrability of a (2+1)-dimensional super nonlinear evolution equation is analyzed in the framework of the fermionic covariant prolongation structure theory. We construct the prolongation structure of the multidimensional super integrable equation and investigate its Lax representation. Furthermore, the Bäcklund transformation is presented and we derive a solution to the super integrable equation.

PACS: 02.30.Ik, 02.30.Jr, 02.40.−k DOI: 10.1088/0256-307X/34/7/070203

The supersymmetric generalizations of integrable equations in 1+1 dimensions have attracted a lot of interest from theoretical physics as well as mathematics, such as the Korteweg–de Vries (KdV) equation,[1,2] the Kadomtsev–Petviashvili (KP) equation,[3] the nonlinear Schrödinger equation (NLSE)[4,5] and the Heisenberg ferromagnet model.[6−8] Supersymmetry offers a powerful tool for widening the scope of integrability of the systems. However, much less is known about the structure and properties of supersymmetric integrable systems in 2+1 dimensions. There has been an increasing interest in the multidimensional supersymmetric integrable equations. Saha and Chowdhury[9] presented two procedures to construct the supersymmetric integrable systems in 2+1 dimensions. Based on the auxiliary matrix variable, an approach to construct (2+1)-dimensional integrable Heisenberg supermagnet (HS) models have been proposed and their gauge equivalent counterparts have been derived.[10] Quite recently, by establishing different auxiliary matrix variables, one constructed two types of (2+1)-dimensional integrable HS models and their integrability has been studied.[11] A number of techniques in standard theory have been extended to investigate the supersymmetric integrable systems, such as the Darboux transformations,[12] the Bäcklund transformations,[13] the Painlevé test,[14] τ functions,[15] the Hirota bilinear method,[16] supertrace identity,[17] binary nonlinearization[18] and prolongation theory.[19]

The prolongation structure theory (PST) proposed by Wahlquist and Estabrook[19] is a very useful method to analyze the nonlinear evolution equation (NEE). There has been considerable interest in the study of integrable systems by means of PST.[20,21] Morris[22,23] generalized this approach to higher dimensions and some (2+1)-dimensional super NEEs have been well investigated. With the successful application of nonlinear realization of connection,[24] Guo *et al.* developed a covariant prolongation structure theory of the NEE. Then Cheng *et al.*[26] established the fermionic covariant PST of (1+1)-dimensional super NEE in terms of the theory of super connection on fibre bundle. Recently, the fermionic covariant PST in the (1+1)-dimensional NEE has been extended to the multidimensional super NEE.[27] In this Letter, we focus our attention on the integrability of (2+1)-dimensional supersymmetric integrable systems in the framework of the multidimensional fermionic covariant PST.

We begin by summarizing the fermionic covariant PST that will be useful in the following. A more detailed description can be found in Ref. [27].

For a (2+1)-dimensional super NEE, it can be transformed into a partial differential equation (PDE) of the first order by adding appropriate new variables. Let us suppose all variables $\{x, y, t, u^\nu, \nu = 4, \ldots, m+3\} \equiv \{x^\mu, \mu = 1, \ldots, m+3\}$, which belong to the super space M with dimension $m+3$. We can represent the corresponding first order PDE as the set of even and odd differential 3-forms,

$$\begin{aligned}&\alpha^{\bar{i}} = dx^\mu \wedge dx^\nu \wedge dx^\gamma h^{\bar{i}}_{\mu\nu\gamma},\\ &(\mu, \nu, \gamma = 1, \ldots, m+3,\ i = 1, \ldots, l),\end{aligned} \tag{1}$$

which constitutes a differential ideal I, i.e., $d\alpha^{\bar{i}} = 0, \bmod(\alpha^1, \ldots, \alpha^l)$. However, these three forms restricted on the solution manifold $S = \{x, y, t, u^\nu(x, y, t)\}$ are null, i.e., $\alpha^{\bar{i}}|_S = 0$, then we obtain the super NEE again.

Now we consider a super principal bundle $P(M, G)$ and a super bundle $E(M, F, G, P)$ with base supermanifold M, fibre F, structure super Lie group G.

The super connection form on E is written as

$$\begin{aligned}\omega^j &= dz^j + dx^\mu \Gamma^j_\mu(x, z)\\ &= dz^j + dx^\mu \Gamma^a_\mu(x)\lambda^j_a(z),\end{aligned} \tag{2}$$

where $x = \{x^\mu, \mu = 1, \ldots, m+3\}$, $z = \{z^i, i = 1, \ldots, n\}$, $\Gamma^a_\mu(x)$ and $\Gamma^j_\mu(x, z)$ are the coefficients of the connection on P and E, respectively. Note that $\Gamma^j_\mu(x, z)$ may be a nonlinear connection, since the variable z in λ^j_a is usually nonlinear.

*Supported by the National Natural Science Foundation of China under Grant Nos 11605096, 11547101 and 11601247.
**Corresponding author. Email: yanzw@imu.edu.cn

Then we introduce the super induced connection $L^j_{k\mu}(x,z(x))$ as follows:[26]

$$L^j_{k\mu}(x,z(x))=\Big[\frac{\partial\lambda^a_k}{\partial x^\mu}+(-1)^{\hat c\hat k}\varGamma^c_\mu\lambda^b_k C^a_{bc}\Big]\lambda^j_a. \quad (3)$$

By means of super induced connection $L_{k\mu}$, we define the following covariant derivative of ω^j restricted on the section,

$$\begin{aligned}D\omega^j&=d\omega^j+\omega^k\wedge L^j_k,\\&=-\frac{1}{2}dx^\nu\wedge dx^\mu(F^a_{\mu\nu}\lambda^j_a)+\frac{1}{2}\omega^l\wedge\omega^k M^j_{kl},\end{aligned} \quad (4)$$

where $L^j_k=dx^\mu L^j_{k\mu}$, $F^a_{\mu\nu}$ and M^j_{kl} are given by

$$F^a_{\mu\nu}=-\frac{\partial\varGamma^a_\nu}{\partial x_\mu}+(-1)^{\hat\mu\hat\nu}\frac{\partial\varGamma^a_\mu}{\partial x_\nu}+(-1)^{(\hat b+\hat\nu)\hat c}\varGamma^c_\mu\varGamma^b_\nu C^a_{cb}, \quad (5)$$

$$M^j_{kl}=(-1)^{\hat l\hat a}\lambda^a_k\frac{\partial\lambda^j_a}{\partial z^l}-(-1)^{\hat k\hat a+\hat l\hat k}\lambda^a_l\frac{\partial\lambda^j_a}{\partial z^k}. \quad (6)$$

One introduces a set of even and odd two forms $\varOmega^j,(j=1,\ldots,k)$ defined on E,

$$\varOmega^j=\beta\wedge\omega^j, \quad (7)$$

where ω^j is given by Eq. (2), and β defined on M is a 1-form to be determined.

Based on Eq. (4), we obtain the covariant derivative of $\varOmega^j$,

$$\begin{aligned}D\varOmega^j&=d\beta\wedge\omega^j-\beta\wedge d\omega^j+\beta\wedge\omega^k\wedge L^j_k\\&=-\frac{1}{2}\beta\wedge dx^\nu\wedge dx^\mu F^j_{\mu\nu}\\&\quad+\Big(d\beta\wedge\omega^j+\frac{1}{2}\varOmega^l\wedge\omega^k M^j_{kl}\Big).\end{aligned} \quad (8)$$

Let us extend the closed ideal I on M to a new closed idea $I'=\{\alpha^{\bar i},\varOmega^j\}$ on E, we have

$$D\omega^j\subset I'. \quad (9)$$

Using Eq. (8) and the closed ideal condition (9), we have

$$\begin{aligned}&-\frac{1}{2}\beta\wedge dx^\nu\wedge dx^\mu F^j_{\mu\nu}+\Big(d\beta\wedge\omega^j\\&+\frac{1}{2}\varOmega^l\wedge\omega^k M^j_{kl}\Big)=\alpha^{\bar i}f^j_{\bar i}+\varOmega^i\wedge\eta^j_i,\end{aligned} \quad (10)$$

where $f^j_{\bar i}$ and η^j_i are the zero and one forms on M, respectively.

Equation (10) can be decomposed into the following two fundamental prolongation structure equations

$$-\frac{1}{2}\beta\wedge dx^\nu\wedge dx^\mu(F^a_{\mu\nu}\lambda^j_a)=\alpha^{\bar i}f^j_{\bar i}, \quad (11)$$

$$\frac{1}{2}\varOmega^l\wedge\omega^k M^j_{kl}=\varOmega^i\wedge\eta^j_i, \quad (12)$$

and the constraint condition

$$d\beta=0. \quad (13)$$

In general, we may completely determine the prolongation structure of a given super nonlinear system when the solutions of the fundamental equation can be found.

Next, we apply the multidimensional fermionic covariant PST to investigate the following super NEE[11]

$$\begin{aligned}&i\varphi_t+\beta\varphi_{xy}+(f_1\varphi)_{xx}-i(f_2\varphi)_x+2f_1(\varphi\bar\varphi+\psi\bar\psi)\varphi\\&+\{\partial_x^{-1}[\beta\partial_y(2\varphi\bar\varphi+\psi\bar\psi)+f_{1x}(2\varphi\bar\varphi+\psi\bar\psi)]\}\varphi\\&+\psi\partial_x^{-1}[\beta\partial_y(\bar\psi\varphi)+f_{1x}\bar\psi\varphi]=0,\\&i\psi_t+\beta\psi_{xy}+(f_1\psi)_{xx}-i(f_2\psi)_x\\&+2f_1\varphi\bar\varphi\psi+\{\partial_x^{-1}[\beta\partial_y(\varphi\bar\varphi)+f_{1x}\varphi\bar\varphi]\}\psi\\&+\varphi\partial_x^{-1}[\beta\partial_y(\bar\varphi\psi)+f_{1x}\bar\varphi\psi]=0,\end{aligned} \quad (14)$$

where φ and ψ are the Grassman even and odd fields, respectively, β is a real constant, and $\partial_x^{-1}f(x,y)$ is the integral of the function $f(x,y)$ with respect to x.

Under the reduction $f_1=f_2=0$ and $\beta=1$, Eq. (14) reduces to the super NLSE[10]

$$\begin{aligned}&i\varphi_t+\varphi_{xy}+[\partial_x^{-1}\partial_y(\varphi\bar\varphi+\psi\bar\psi)]\varphi+\varphi\partial_x^{-1}\partial_y(\bar\varphi\varphi)\\&+\psi\partial_x^{-1}\partial_y(\bar\psi\varphi)=0,\\&i\psi_t+\psi_{xy}+[\partial_x^{-1}\partial_y(\varphi\bar\varphi)]\psi+\varphi\partial_x^{-1}\partial_y(\bar\varphi\psi)=0.\end{aligned} \quad (15)$$

Let us consider the integrability of Eq. (14) by means of the (2+1)-dimensional fermionic covariant PST. Taking $\varphi_1=\varphi_y$, $\bar\varphi_1=\bar\varphi_y$, $\psi_1=\psi_y,\bar\psi_1=\bar\psi_y$, $k=\varphi_x$, $\bar k=\bar\varphi_x$, $l=\psi_x$, $\bar l=\bar\psi_x$, $m=\partial_x^{-1}(\varphi\bar\varphi)$, $n=\partial_x^{-1}(\psi\bar\psi)$, $p=\partial_x^{-1}\partial_y(\varphi\bar\varphi)$, $q=\partial_x^{-1}\partial_y(\bar\psi\varphi)$, $r=\partial_x^{-1}\partial_y(\bar\varphi\psi)$, $s=\partial_x^{-1}\partial_y(\psi\bar\psi)$, $u=\partial_x^{-1}(\bar\psi\varphi)$ and $v=\partial_x^{-1}(\bar\varphi\psi)$ as new independent variables, we can define the 3-forms in the twenty five-dimensional space $M=\{t,x,y,\varphi,\bar\varphi,\psi,\bar\psi,\varphi_1,\bar\varphi_1,\psi_1,\bar\psi_1,k,\bar k,l,\bar l,m,n,p,q,r,s,u,v,f_1,f_2\}$,

$$\begin{aligned}\alpha_1&=dt\wedge dx\wedge d\varphi-\varphi_1dt\wedge dx\wedge dy,\\\alpha_2&=dt\wedge dx\wedge d\bar\varphi-\bar\varphi_1dt\wedge dx\wedge dy,\\\alpha_3&=dt\wedge dx\wedge d\psi-\psi_1dt\wedge dx\wedge dy,\\\alpha_4&=dt\wedge dx\wedge d\bar\psi-\bar\psi_1dt\wedge dx\wedge dy,\\\alpha_5&=dt\wedge d\varphi\wedge dy-kdt\wedge dx\wedge dy,\\\alpha_6&=dt\wedge d\bar\varphi\wedge dy-\bar kdt\wedge dx\wedge dy,\\\alpha_7&=dt\wedge d\psi\wedge d\psi-ldt\wedge dx\wedge dy,\\\alpha_8&=dt\wedge d\bar\psi\wedge dy-\bar ldt\wedge dx\wedge dy,\\\alpha_9&=dt\wedge dp\wedge dy-p_xdt\wedge dx\wedge dy,\\\alpha_{10}&=dt\wedge dq\wedge dy-q_xdt\wedge dx\wedge dy,\\\alpha_{11}&=dt\wedge dr\wedge dy-r_xdt\wedge dx\wedge dy,\\\alpha_{12}&=dt\wedge ds\wedge dy-s_xdt\wedge dx\wedge dy,\\\alpha_{13}&=dt\wedge du\wedge dy-u_xdt\wedge dx\wedge dy,\\\alpha_{14}&=dt\wedge dv\wedge dy-v_xdt\wedge dx\wedge dy,\\\alpha_{15}&=id\varphi\wedge dx\wedge dy+\beta dt\wedge d\varphi_1\wedge dy\\&\quad+f_1dt\wedge dk\wedge dy+Adt\wedge dx\wedge dy,\\\alpha_{16}&=id\psi\wedge dx\wedge dy+\beta dt\wedge d\psi_1\wedge dy\\&\quad+f_1dt\wedge dl\wedge dy+Cdt\wedge dx\wedge dy,\\\alpha_{17}&=id\bar\varphi\wedge dx\wedge dy-\beta dt\wedge d\bar\varphi_1\wedge dy\\&\quad-f_1dt\wedge d\bar k\wedge dy-Bdt\wedge dx\wedge dy,\end{aligned}$$

$$\begin{aligned}\alpha_{18} &= id\bar{\psi}\wedge dx\wedge dy-\beta dt\wedge d\bar{\psi}_1\wedge dy\\ &\quad -f_1 dt\wedge d\bar{l}\wedge dy-Ddt\wedge dx\wedge dy,\\ \alpha_{19} &= dt\wedge dm\wedge dy-m_x dt\wedge dx\wedge dy,\\ \alpha_{20} &= dt\wedge dn\wedge dy-n_x dt\wedge dx\wedge dy,\\ \alpha_{21} &= dt\wedge df_1\wedge dy-f_{1x}dt\wedge dx\wedge dy,\\ \alpha_{22} &= dt\wedge df_2\wedge dy-f_{2x}dt\wedge dx\wedge dy, \end{aligned}\tag{16}$$

where A, B, C and D are as follows:

$$\begin{aligned}A &= 2f_{1x}k-if_{2x}\varphi-if_2k+\beta[(2p+s)\varphi+\psi q]\\ &\quad +2f_1(\varphi\bar{\varphi}+\psi\bar{\psi})\varphi+f_{1x}[(2m+n)\varphi+\psi u],\\ B &= 2f_{1x}\bar{k}+if_{2x}\bar{\varphi}+if_2\bar{k}+\beta[\bar{\varphi}(2p+s)+r\bar{\psi}]\\ &\quad +2f_1\bar{\varphi}(\varphi\bar{\varphi}+\psi\bar{\psi})+f_{1x}[(2m+n)\bar{\varphi}+v\bar{\psi}],\\ C &= 2f_{1x}l-if_{2x}\psi-if_2l+\beta(p\psi+\varphi r)\\ &\quad +2f_1\varphi\bar{\varphi}\psi+f_{1x}(\varphi v+m\psi),\\ D &= 2f_{1x}\bar{l}+if_{2x}\bar{\psi}+if_2\bar{l}+\beta(\bar{\psi}p+q\bar{\varphi})\\ &\quad +2f_1\varphi\bar{\varphi}\bar{\psi}+f_{1x}(u\bar{\varphi}+\bar{\psi}m). \end{aligned}\tag{17}$$

To establish the fermionic covariant prolongation structure, we extend the above ideal I by adding to it a set of even and odd two forms $\Omega^j,(j=1,\ldots,k)$,

$$\begin{aligned}\Omega^j &= \beta\wedge\omega^j\\ &= \beta\wedge(dz^j+dx^\mu\Gamma^j_\mu(X,z)),\\ &\quad j=1,\ldots,p,p+1,\ldots,q,\end{aligned}\tag{18}$$

where β defined on M is a 1-form to be determined, $M=\{x^\mu,\mu=1,\ldots,25\}=\{t,x,y,\varphi,\bar{\varphi},\psi,\bar{\psi},\psi_1,\bar{\psi}_1,\psi_1,\bar{\psi}_1,k,l,m,n,p,q,r,s,u,v,f_1,f_2\}$. According to the multidimensional fermionic covariant PST developed, the closed condition of the extended ideal will lead to the covariant fundamental Eqs. (11), (12) and the constraint condition (13).

Then we obtain its prolongation structure when the solutions of one fundamental equation can be found.

From Eq. (13), we suppose

$$\beta = C_\mu dx^\mu,\tag{19}$$

where C_μ is a constant. By substituting the 3-form Eq. (16) and 1-form Eq. (19) into the fundamental Eq. (11), we obtain

$$\begin{aligned}&C_1F^j_{26}-C_2F^j_{16}+C_6F^j_{12}=0,\\ &C_1F^j_{27}-C_2F^j_{17}+C_7F^j_{12}=0,\\ &C_1F^j_{210}-C_2F^j_{110}+C_{10}F^j_{12}=0,\\ &C_1F^j_{211}-C_2F^j_{111}+C_{11}F^j_{12}=0,\\ &C_1F^j_{212}-C_2F^j_{112}+C_{12}F^j_{12}=0,\\ &C_1F^j_{213}-C_2F^j_{113}+C_{13}F^j_{12}=0,\\ &C_1F^j_{214}-C_2F^j_{114}+C_{14}F^j_{12}=0,\\ &C_1F^j_{215}-C_2F^j_{115}+C_{15}F^j_{12}=0,\\ &C_1F^j_{216}-C_2F^j_{116}+C_{16}F^j_{12}=0,\\ &C_1F^j_{217}-C_2F^j_{117}+C_{17}F^j_{12}=0,\\ &C_1F^j_{218}-C_2F^j_{118}+C_{18}F^j_{12}=0,\\ &C_1F^j_{219}-C_2F^j_{119}+C_{19}F^j_{12}=0,\\ &C_1F^j_{220}-C_2F^j_{120}+C_{20}F^j_{12}=0,\\ &C_1F^j_{221}-C_2F^j_{121}+C_{21}F^j_{12}=0,\\ &C_1F^j_{222}-C_2F^j_{122}+C_{22}F^j_{12}=0,\\ &C_1F^j_{223}-C_2F^j_{123}+C_{23}F^j_{12}=0,\\ &C_1F^j_{224}-C_2F^j_{124}+C_{243}F^j_{12}=0,\\ &C_1F^j_{225}-C_2F^j_{125}+C_{25}F^j_{12}=0,\\ &C_1F^j_{312}-C_3F^j_{112}-if_1(C_2F^j_{34}\\ &-C_3F^j_{24})+C_{12}F^j_{13}=0,\\ &C_1F^j_{313}-C_3F^j_{113}+if_1(C_2F^j_{35}\\ &-C_3F^j_{25})+C_{13}F^j_{13}=0,\\ &C_1F^j_{314}-C_3F^j_{114}-if_1(C_2F^j_{38}\\ &-C_3F^j_{28})+C_{14}F^j_{13}=0,\\ &C_1F^j_{315}-C_3F^j_{115}+if_1(C_2F^j_{39}\\ &-C_3F^j_{29})+C_{15}F^j_{13}=0,\\ &C_1F^j_{316}-C_3F^j_{116}=0,\quad C_1F^j_{317}-C_3F^j_{117}=0\\ &C_2F^j_{36}-C_3F^j_{26}+C_6F^j_{23}=0,\\ &C_2F^j_{37}-C_3F^j_{27}+C_7F^j_{23}=0,\\ &C_2F^j_{310}-C_3F^j_{210}+C_{10}F^j_{23}=0,\\ &C_2F^j_{311}-C_3F^j_{211}+C_{11}F^j_{23}=0,\\ &C_2F^j_{312}-C_3F^j_{212}+C_{12}F^j_{23}=0,\\ &C_2F^j_{313}-C_3F^j_{213}+C_{13}F^j_{23}=0,\\ &C_2F^j_{314}-C_3F^j_{214}+C_{14}F^j_{23}=0,\\ &C_2F^j_{315}-C_3F^j_{215}+C_{15}F^j_{23}=0,\\ &C_2F^j_{316}-C_3F^j_{216}+C_{16}F^j_{23}=0,\\ &C_2F^j_{317}-C_3F^j_{217}+C_{17}F^j_{23}=0,\\ &C_2F^j_{318}-C_3F^j_{218}+C_{18}F^j_{23}=0,\\ &C_2F^j_{319}-C_3F^j_{219}+C_{19}F^j_{23}=0,\\ &C_2F^j_{320}-C_3F^j_{220}+C_{20}F^j_{23}=0,\\ &C_2F^j_{321}-C_3F^j_{221}+C_{21}F^j_{23}=0,\\ &C_2F^j_{322}-C_3F^j_{222}+C_{22}F^j_{23}=0,\\ &C_2F^j_{323}-C_3F^j_{223}+C_{23}F^j_{23}=0,\\ &C_2F^j_{324}-C_3F^j_{224}+C_{24}F^j_{23}=0,\\ &C_2F^j_{325}-C_3F^j_{225}+C_{25}F^j_{23}=0,\\ &\frac{f_1}{\beta}(C_3F^j_{16}-C_1F^j_{36})+C_1F^j_{312}\\ &-C_3F^j_{112}+C_{12}F^j_{13}=0,\\ &\frac{f_1}{\beta}(C_3F^j_{17}-C_1F^j_{37})+C_1F^j_{313}\\ &-C_3F^j_{113}+C_{13}F^j_{13}=0,\\ &\frac{f_1}{\beta}(C_3F^j_{110}-C_1F^j_{310})+C_1F^j_{314}\\ &-C_3F^j_{114}+C_{14}F^j_{13}=0,\end{aligned}$$

$$\frac{f_1}{\beta}(C_3F^j_{111}-C_3F^j_{311})+C_1F^j_{315}$$
$$-C_3F^j_{115}+C_{15}F^j_{13}=0,$$
$$\frac{i}{\beta}(C_3F^j_{16}-C_1F^j_{36})+C_3F^j_{24}-C_2F^j_{34}-C_4F^j_{23}=0,$$
$$\frac{i}{\beta}(C_1F^j_{37}-C_3F^j_{17})+C_3F^j_{25}-C_2F^j_{35}-C_5F^j_{23}=0,$$
$$\frac{i}{\beta}(C_3F^j_{110}-C_1F^j_{310})+C_3F^j_{28}-C_2F^j_{38}-C_8F^j_{23}=0,$$
$$\frac{i}{\beta}(C_3F^j_{111}-C_1F^j_{111})+C_3F^j_{29}-C_2F^j_{39}-C_9F^j_{23}=0,$$
$$-C_1F^j_{23}-\varphi_1(C_1F^j_{24}-C_2F^j_{14})-\bar{\varphi}_1(C_1F^j_{25}-C_2F^j_{15})$$
$$-\psi_1(C_1F^j_{28}-C_2F^j_{18})-\bar{\psi}_1(C_1F^j_{29}-C_2F^j_{19})$$
$$+k(C_1F^j_{34}-C_3F^j_{14})$$
$$+\bar{k}(C_1F^j_{35}-C_3F^j_{15})-\frac{1}{\beta}[A(C_1F^j_{36}-C_3F^j_{16})$$
$$+B(C_1F^j_{37}-C_3F^j_{17})+C(C_1F^j_{310}-C_3F^j_{110})$$
$$+D(C_1F^j_{311}-C_3F^j_{111})]-\psi_1(C_1F^j_{28}-C_2F^j_{18})$$
$$+l(C_1F^j_{38}-C_3F^j_{18})+\bar{l}(C_1F^j_{39}-C_3F^j_{19})$$
$$+p_x(C_1F^j_{318}-C_3F^j_{118})+q_x(C_1F^j_{319}-C_3F^j_{119})$$
$$+r_x(C_1F^j_{320}-C_3F^j_{120})+s_x(C_1F^j_{321}-C_3F^j_{121})$$
$$+u_x(C_1F^j_{322}-C_3F^j_{122})+v_x(C_1F^j_{323}-C_3F^j_{123})$$
$$+C_2F^j_{13}-C_3F^j_{12}+m_x(C_1F^j_{316}-C_3F^j_{116})$$
$$+n_x(C_1F^j_{317}-C_3F^j_{117})+f_{1x}(C_1F^j_{324}-C_3F^j_{124})$$
$$+f_{2x}(C_1F^j_{325}-C_3F^j_{125})=0. \quad (20)$$

Solving Eq. (20), we have the constants C_μ and connection coefficients as follows:

$$C_1=1,\ \ C_2=0,\ \ C_3=-\frac{1}{\lambda\beta},\ \ C_\mu=0\ (\mu\geqslant 4),$$
$$\varGamma^a_1=0,\ \varGamma^a_\mu=0\ (\mu\geqslant 4,\ a=1,\dots 8,\ \mu=1,\dots 25),$$
$$\varGamma^1_2=\frac{i}{2}(\varphi+\bar{\varphi}),\ \varGamma^2_2=\frac{i}{2}(\varphi-\bar{\varphi}),\ \varGamma^3_2=-\frac{i\lambda}{2},\ \varGamma^4_2=\frac{i\lambda}{2},$$
$$\varGamma^5_2=\frac{i}{2}(\psi+\bar{\psi}),\ \varGamma^6_2=\frac{1}{2}(\psi-\bar{\psi}),\ \varGamma^7_2=0,\ \ \varGamma^8_2=0,$$
$$\varGamma^1_3=\frac{1}{2\lambda\beta}[\beta(\bar{\varphi}_1-\varphi_1)+f_{1x}(\bar{\varphi}-\varphi)+f_1(\bar{k}-k)$$
$$+i(f_2-\lambda f_1)(\varphi+\bar{\varphi})],$$
$$\varGamma^2_3=\frac{i}{2\lambda\beta}[\beta(\bar{\varphi}_1+\varphi_1)+f_{1x}(\bar{\varphi}+\varphi)+f_1(\bar{k}+k)$$
$$+i(f_1-\lambda f_2)(\varphi-\bar{\varphi})],$$
$$\varGamma^3_3=\frac{i}{2\lambda\beta}[\lambda^2 f_1-\lambda f_2-\beta(2p+s)-f_1(2\varphi\bar{\varphi}+\psi\bar{\psi})$$
$$-f_{1x}(2m+n)],$$
$$\varGamma^4_3=\frac{i}{2\lambda\beta}(-\lambda^2 f_1+\lambda f_2-\beta s-f_1\psi\bar{\psi}-f_{1x}n),$$
$$\varGamma^5_3=\frac{1}{2\lambda\beta}[\beta(\bar{\psi}_1-\psi_1)+f_{1x}(\bar{\psi}-\psi)+f_1(\bar{l}-l)$$
$$+i(f_2-\lambda f_1)(\bar{\psi}+\psi)],$$
$$\varGamma^6_3=\frac{i}{2\lambda\beta}[\beta(\bar{\psi}_1+\psi_1)+f_{1x}(\bar{\psi}+\psi)+f_1(\bar{l}+l)$$
$$+i(f_2-\lambda f_1)(\bar{\psi}-\psi)],$$
$$\varGamma^7_3=\frac{i}{2\lambda\beta}[\beta(r+q)+f_{1x}(u+v)+f_1(\bar{\varphi}\psi+\bar{\psi}\varphi)],$$
$$\varGamma^8_3=\frac{1}{2\lambda\beta}[\beta(r-q)+f_{1x}(v-u)+f_1(\bar{\varphi}\psi-\bar{\psi}\varphi)]. \quad (21)$$

The prolongation algebra is $su(2/1)\times R(\lambda)$, where the parameter λ is a complex constant. Its commutation relations are given by

$$\begin{aligned}
&[T_1,T_2]=2i\lambda T_3,\ [T_1,T_3]=-2i\lambda T_2,\\
&[T_1,T_4]=0,\ [T_1,T_5]=i\lambda T_8,\\
&[T_1,T_6]=-i\lambda T_7,\ [T_1,T_7]=i\lambda T_6,\\
&[T_1,T_8]=-i\lambda T_5,\ [T_2,T_3]=2i\lambda T_1,\\
&[T_2,T_4]=0,\ [T_2,T_5]=i\lambda T_7,\\
&[T_2,T_6]=i\lambda T_8,\ [T_2,T_7]=-i\lambda T_5,\\
&[T_2,T_8]=-i\lambda T_6,\ [T_3,T_4]=0,\\
&[T_3,T_5]=i\lambda T_6,\ [T_3,T_6]=i\lambda T_5,\\
&[T_3,T_7]=-i\lambda T_8,\ [T_3,T_8]=i\lambda T_7,\\
&[T_4,T_5]=-i\lambda T_6,\ [T_4,T_6]=i\lambda T_5,\\
&[T_4,T_7]=-i\lambda T_8,\ [T_4,T_8]=i\lambda T_7,\\
&[T_5,T_5]_+=\lambda^3(T_3+T_4),[T_5,T_6]_+=0,\\
&[T_5,T_7]_+=\lambda^3 T_1,\ [T_5,T_8]_+=-\lambda^3 T_2,\\
&[T_6,T_6]_+=\lambda^3(T_3+T_4),[T_6,T_7]_+=\lambda^3 T_2,\\
&[T_6,T_8]_+=\lambda^3 T_1,\ [T_7,T_7]_+=\lambda^3(T_4-T_3),\\
&[T_7,T_8]_+=0,\ [T_8,T_8]_+=\lambda^3(T_4-T_3),
\end{aligned} \quad (22)$$

where T_i's, for $i=1,\dots,4$ and $i=5,\dots,8$, are bosonic and fermionic generators, respectively, and $[T_i,T_j]_+$ denotes the anti commutation relations. We derive a linear realization of the prolongation algebra as follows:

$$\begin{aligned}
&T_1=\lambda z_2\frac{\partial}{\partial z_1}+\lambda z_1\frac{\partial}{\partial z_2},\ T_2=i\lambda z_2\frac{\partial}{\partial z_1}-i\lambda z_1\frac{\partial}{\partial z_2},\\
&T_3=\lambda z_1\frac{\partial}{\partial z_1}-\lambda z_2\frac{\partial}{\partial z_2},\\
&T_4=\lambda z_1\frac{\partial}{\partial z_1}+\lambda z_2\frac{\partial}{\partial z_2}+2\lambda\xi\frac{\partial}{\partial \xi},\\
&T_5=\lambda^2\xi\frac{\partial}{\partial z_1}-i\lambda^2 z_1\frac{\partial}{\partial \xi},\ T_6=i\lambda^2\xi\frac{\partial}{\partial z_1}-i\lambda^2 z_1\frac{\partial}{\partial \xi},\\
&T_7=\lambda^2\xi\frac{\partial}{\partial z_1}-i\lambda^2 z_1\frac{\partial}{\partial \xi},\ T_8=i\lambda^2\xi\frac{\partial}{\partial z_2}-i\lambda^2 z_2\frac{\partial}{\partial \xi},
\end{aligned} \quad (23)$$

where z_1 and z_2 are even prolongation variables. We denote odd prolongation variable z_3 by ξ.

By requiring $\varOmega^j|_S=0$, it yields the inverse scattering equation

$$\begin{pmatrix} z_1 \\ z_2 \\ \xi \end{pmatrix}_x = -i\begin{pmatrix} 0 & \varphi & \psi \\ \bar{\varphi} & \lambda & 0 \\ \bar{\psi} & 0 & \lambda \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ \xi \end{pmatrix},$$
$$\begin{pmatrix} z_1 \\ z_2 \\ \xi \end{pmatrix}_t = -\lambda\beta\begin{pmatrix} z_1 \\ z_2 \\ \xi \end{pmatrix}_y + G\begin{pmatrix} z_1 \\ z_2 \\ \xi \end{pmatrix}, \quad (24)$$

where G is

$$\tilde{G} = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}, \quad (25)$$

with

$$\begin{aligned} G_{11} =& i\partial_x^{-1}[\beta\partial_y(\varphi\bar{\varphi}+\psi\bar{\psi}) + 2f_{1x}(\varphi\bar{\varphi}+\psi\bar{\psi}) \\ &+ f_1\partial_x(\varphi\bar{\varphi}+\psi\bar{\psi})] \\ G_{12} =& \beta\varphi_y + \partial_x(f_1\varphi) - f_2\varphi + i\lambda f_1\varphi, \\ G_{13} =& \beta\psi_y + \partial_x(f_1\psi) - f_2\psi + i\lambda f_1\psi, \\ G_{21} =& -\beta\bar{\varphi}_y - \partial_x(f_1\bar{\varphi}) - f_2\bar{\varphi} + i\lambda f_1\bar{\varphi}, \\ G_{22} =& -i\partial_x^{-1}[\beta\partial_y(\varphi\bar{\varphi}) + 2f_{1x}(\varphi\bar{\varphi}) + f_1\partial_x(\varphi\bar{\varphi})] \\ &+ i\lambda^2 f_1 - i\lambda f_2, \\ G_{23} =& -i\partial_x^{-1}[\beta\partial_y(\bar{\varphi}\psi) + 2f_{1x}(\bar{\varphi}\psi) + f_1\partial_x(\bar{\varphi}\psi)], \quad (26) \\ G_{31} =& -\beta\bar{\psi}_y - \partial_x(f_1\bar{\psi}) - f_2\bar{\psi} + i\lambda f_1\bar{\psi}, \\ G_{32} =& -i\partial_x^{-1}[\beta\partial_y(\bar{\psi}\varphi) + 2f_{1x}(\bar{\psi}\varphi) + f_1\partial_x(\bar{\psi}\varphi)], \\ G_{33} =& -i\partial_x^{-1}[\beta\partial_y(\bar{\psi}\psi) + 2f_{1x}(\bar{\psi}\psi) + f_1\partial_x(\bar{\psi}\psi)] \\ &+ i\lambda^2 f_1 - i\lambda f_2. \quad (27) \end{aligned}$$

The super Riccati equation and the Bäcklund transformation of the $(2+1)$-dimensional super NLSE have been investigated in Ref. [11] and the solution of Eq. (14) has been given as follows:

$$\begin{aligned} \varphi' =& \Big\{(\bar{\alpha}_1 - \alpha_1)\xi \exp\Big[\frac{if_1(y+\alpha_1)^2}{t\beta^2} \\ &- \frac{i(y+\alpha_1)x + if_2ty}{t\beta}\Big]\Big\}\Big/\Big\{t\beta(1+|\beta|^2 \\ &\cdot \exp\Big[\frac{i(\alpha_1-\bar{\alpha}_1)(2f_1y+x\beta) - 2if_2\beta ty}{t\beta^2}\Big] \\ &- \theta\bar{\theta}|\gamma|^2)\Big\}, \\ \psi' =& \Big\{(\bar{\alpha}_1 - \alpha_1)\xi \exp\Big[\frac{if_1(y+\alpha_1)^2}{t\beta^2} \\ &- \frac{i(y+\alpha_1)x + if_2ty}{t\beta}\Big]\bar{\theta}\bar{\gamma}\Big\}\Big/\Big\{t\beta(1+|\beta|^2 \\ &\cdot \exp\Big[\frac{i(\alpha_1-\bar{\alpha}_1)(2f_1y+x\beta) - 2if_2\beta ty}{t\beta^2}\Big] \\ &- \theta\bar{\theta}|\gamma|^2)\Big\}, \quad (28) \end{aligned}$$

where the parameters α and γ are the Grassmann even constants, and θ is the Grassmann odd constant.

In summary, we have investigated the integrability of the (2+1)-dimensional super NEE by means of the multidimensional fermionic covariant PST, where the $su(2/1)\times R(\lambda)$ prolongation structure of the super NEE has been presented. The Lax representation of the super NEE has been studied in terms of the representations of the prolongation algebra. Moreover, a solution to the (2+1)-dimensional super NEE is derived. It is pointed out that the conservation laws can be constructed by means of the Lax pair and the Riccati equation for the integrable systems.[28] How to construct the conservation laws based on the super Riccati equation for the super integrable systems is still under investigation. As to the multidimensional super NEEs in this study, their applications should be of interest.

References

[1] Mathieu P 1988 *J. Math. Phys.* **29** 2499
[2] Bellucci S, Ivanov E, Krivonos S and Pichugin A 1993 *Phys. Lett.* B **312** 463
[3] Manin Yu I and Radul A O 1985 *Commun. Math. Phys.* **98** 65
[4] Roelofs G H M and Kersten P H M 1992 *J. Math. Phys.* **33** 2185
[5] Popowicz Z 1994 *Phys. Lett.* A **194** 375
[6] Makhankov V G and Pashaev O K 1992 *J. Math. Phys.* **33** 2923
[7] Guo J F, Wang S K, Wu K, Yan Z W and Zhao W Z 2009 *J. Math. Phys.* **50** 113502
[8] Yan Z W, Li M L, Wu K and Zhao W Z 2010 *Commun. Theor. Phys.* **53** 21
[9] Saha M and Roy C A 1999 *Int. J. Theor. Phys.* **38** 2037
[10] Yan Z W, Chen M R, Wu K and Zhao W Z 2012 *J. Phys. Soc. Jpn.* **81** 094006
[11] Yan Z W 2017 *Z. Naturforsch.* A **72** 331
[12] Liu Q P 1995 *Lett. Math. Phys.* **35** 115
[13] Chaichian M and Kulish P P 1978 *Phys. Lett.* B **78** 413
[14] Mathieu P 1988 *Phys. Lett.* A **128** 169
[15] Ibort A, Martinez Alonso L and Reus E 1996 *J. Math. Phys.* **37** 6157
[16] McArthur I N and Yung C M 1993 *Mod. Phys. Lett.* A **8** 1739
[17] Ma W X, He J S and Qin Z Y 2008 *J. Math. Phys.* **49** 033511
[18] Yu J, He J S, Ma W X and Cheng Y 2010 *Chin. Ann. Math.* B **31** 361
[19] Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1
[20] Wang D S, Yin S J, Tian Y and Liu Y F 2014 *Appl. Math. Comput.* **229** 296
[21] Wang D S and Wei X Q 2016 *Appl. Math. Lett.* **51** 60
[22] Morris H C 1976 *J. Math. Phys.* **17** 1870
[23] Morris H C 1977 *J. Math. Phys.* **18** 285
[24] Lu Q K, Guo H Y and Wu K 1983 *Commun. Theor. Phys.* **2** 1029
[25] Guo H Y, Hsiang Y Y and Wu K 1982 *Commun. Theor. Phys.* **1** 495
[26] Cheng J P, Wang S K, Wu K and Zhao W Z 2010 *J. Math. Phys.* **51** 093501
[27] Yan Z W, Li M L, Wu K and Zhao W Z 2013 *J. Math. Phys.* **54** 033506
[28] Yang J Y and Ma W X 2016 *Mod. Phys. Lett.* B **30** 1650381