

Valid Regions of Formulas of Sound Speed in Bubbly Liquids *

Yu-Ning Zhang(张宇宁)**, Zhong-Yu Guo(郭忠玉), Yu-Hang Gao(高宇航), Xiao-Ze Du(杜小泽)

Key Laboratory of Condition Monitoring and Control for Power Plant Equipment (Ministry of Education),
North China Electric Power University, Beijing 102206

(Received 16 January 2017)

There are numerous formulae relating to the predictions of sound wave in the cavitating and bubbly flows. However, the valid regions of those formulae are rather unclear from the view point of physics. In this work, the validity of the existing formulae is discussed in terms of three regions by employing the analysis of three typical lengths involved (viscous length, thermal diffusion length and bubble radius). In our discussions, viscosity and thermal diffusion are both considered together with the effects of relative motion between bubbles and liquids. The importance of relative motion and thermal diffusion are quantitatively discussed in a wide range of parameter zones (including bubble radius and acoustic frequency). The results show that for large bubbles, the effects of relative motion will be prominent in a wide region.

PACS: 47.55.dd, 43.35.Ei, 62.60.+v

DOI: 10.1088/0256-307X/34/6/064701

Bubbly liquids (containing both bubbles and water) widely exist in cavitating flows inside hydroturbines,^[1–4] underwater explosions^[5,6] and ventilated flows for drag reduction.^[7,8] Sound speeds in those bubbly liquids are of great importance to researchers for modelling the above complex flow^[9,10] or further engineering parameter optimization. Even with only a very limited volume fraction in the liquids, bubbles could significantly affect the sound speeds to a great extent. Hence, a detailed study of the wave propagation in bubbly liquids is essential to both the fundamental research of cavitation and its applications.

Various kinds of formulae have been proposed.^[11–20] For bubbly flows with large void fraction, there are numerous formulae involved, including Wood,^[13] Brennen,^[14] and Crespo.^[17] For the case with small void fraction, some simplified formulae could be obtained, e.g., Ando *et al.*^[15] and Prosperetti.^[16] Zhang and Du^[18] considered the influences of non-uniformity outside bubbles on the wave propagations. In a recent work, Zhang *et al.*^[19] further analyzed the wave propagation in complex vapor-gas mixture bubbles and the frequency-response curves are categorized into two groups according to the position of the minimum wave speed. One of the current barriers in this field is that there is no complete discussion on the valid regions of the above formulae from the fundamental physical point of view. There are two important effects involved in the sound wave propagation in bubbly liquids: viscous effect and thermal diffusion effect. A detailed analysis of the above effects in a full parameter zone (e.g., frequencies) could provide profound information on understanding of the sound speed in bubbly liquids.

In this Letter, a detailed analysis is performed to shed light on the effects of viscous and thermal diffusion on the sound speed in bubbly liquids. Furthermore, the full parameter zone is divided into three regions by comparing three typical characteristic lengths (viscous length, thermal diffusion length and bubble

radius) involved to quantitatively discuss the influences of viscous and thermal diffusion effects. The valid regions of typical existing formulae are given in the range of whole void fractions. In addition, some simplified formulae of sound speeds at low frequencies are also theoretically discussed and compared.

First, let us introduce the theoretical model of propagation of sound waves in bubble liquids. In this model, the mixture of bubble/water is assumed to be a homogeneous fluid. The bubbles are assumed to be spherical ones because the non-spherical shapes of bubbles usually appear near the wall^[21,22] or with unstable surface oscillations.^[23] The sound wave is assumed to be an acoustic wave with a simple single frequency. For the cases with more complex wave patterns, we can refer to Zhang *et al.*^[24–26] Now, it is further assumed that the bubbles in the mixture are monodisperse. From the thermodynamic point of view, the speed of sound (c_m) is defined as

$$c_m = \left(\frac{dp}{d\rho_m} \right)^{\frac{1}{2}},$$

where p is the pressure in the bubbly liquids, and ρ_m is the averaged density of the bubbly mixture. Hence, ρ_m could be expressed as

$$\rho_m = \rho_g \beta + \rho_l (1 - \beta), \quad (1)$$

where ρ_g and ρ_l denote the densities of the gas bubble and liquid, respectively; β is defined as void fraction representing the percentage of gas content (in terms of volume) in unit volume mixture. By introducing the surface tension (σ), the equilibrium pressure in the bubble (p_G) can be expressed as $p_G = p_0 + 2\sigma/R_0$. Here p_0 is the ambient pressure, R_0 is the equilibrium bubble radius. Therefore, the sound speed in the mixture (c_m) can be expressed as^[11]

$$\begin{aligned} \frac{1}{c_m^2} = \frac{d\rho_m}{dp} &= (1 - \beta) \frac{d\rho_l}{dp} + \beta \frac{d\rho_g}{dp} \\ &+ (\rho_g - \rho_l) \frac{d\beta}{dp}. \end{aligned} \quad (2)$$

*Supported by the National Natural Science Foundation of China under Grant No 51506051, the National Basic Research Program of China under Grant No 2015CB251503, and the Fundamental Research Funds for the Central Universities under Grant No JB2015RCY04.

**Corresponding author. Email: y.zhang@ncepu.edu.cn

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According to the definition of sound speed, one can define

$$c_l^2 = \frac{dp}{d\rho_l}, \quad c_g^2 = \frac{dp}{d\rho_g},$$

where c_l is the sound speed in the (pure) liquid, and c_g is the sound speed in the (pure) gas. Noticing that $(\rho_g c_g)^2$ is far less than $(\rho_l c_l)^2$, Eq. (2) can be simplified to^[11,13]

$$\frac{1}{c_m^2} = \frac{(1-\beta)^2}{c_l^2} + \frac{\beta^2}{c_g^2} + \frac{\rho_l \beta(1-\beta)}{p}. \quad (3)$$

This equation was initially derived by Wood.^[13] Generally, as long as the value of β is not very close to zero or unity, the first and second terms on the right-hand side of Eq. (3) can be ignored. Hence, Eq. (3) can be reduced to^[13]

$$c_m^2 = \frac{p_G}{\rho_l \beta(1-\beta)}, \quad (4)$$

where the surface tension (referring to p_G) is considered in Eq. (4) based on the classical Wood formula. In this equation, the sound speed is dependent on the bubble radius (R_0) through the p_G term, while it is independent of the sound frequency (ω). This equation is widely used in the calculation of sound speed in the homogeneous bubbly mixtures.

Later, Crespo^[17] proposed some revisions based on Eq. (4), which could be generalized as

$$c_m^2 = \frac{\alpha \kappa p_G}{\rho_l \beta(1-\beta)}, \quad (5)$$

where α is the coefficient depending on the relative sizes of viscous length, thermal diffusion length and bubble radius; κ is the polytropic exponent, reflecting the thermodynamic status inside bubbles. For example, $\kappa = 1$ represents the isothermal status while $\kappa = \gamma$ (γ is the ratio of the specific heats) represents the adiabatic status. For the regions between the above two extremes, κ varies between 1 and γ . Strictly speaking, κ depends on the non-dimensional parameter G ($G = \omega R_0^2 / D_g$). For a large G , the pressure inside the bubble will be significantly non-uniform, leading to a dramatic decrease of the value of κ in terms of an effective parameter. For a detailed calculation of κ , researchers can refer to Zhang.^[27]

Now, several typical lengths involved in this phenomenon will be discussed to determine the effects of viscosity and thermal diffusion. For more details, researchers are referred to van Wijngaarden^[11] and Crespo.^[17] The viscous length (defined as δ_v) is of the order of $\sqrt{\nu/\omega}$, where ν and ω are the kinematic viscosity of liquid and the frequency of the external excitation in the liquid, respectively. The thermal diffusion length (defined as δ_{th}) is of the order of $\sqrt{D_g/\omega}$, where D_g is the thermal diffusivity of the gas inside the bubble. Another typical length is the bubble radius. As in water, ν is always smaller than D_g . Hence δ_v is also smaller than δ_{th} . Based on the expressions of those typical lengths, one can find that the paramount parameters are the bubble radius and the frequency,

which finally determine the values of those three typical lengths. Quantitative comparisons of the above three typical lengths are given in the following.

In regions with low frequency, the three typical lengths satisfy

$$R_0 \ll \sqrt{\frac{\nu}{\omega}} \ll \sqrt{\frac{D_g}{\omega}}.$$

The viscosity has a great influence on wave propagation by reducing the velocity differences between bubbles and liquids. In other words, there is no relative motion between two phases and the bubbles travel with the same speed as the liquids. As the thermal diffusion length is much larger than the bubble radius, prominent heat transfer exists over the whole bubble interface and the temperatures of bubbles and liquids across the bubble surface show little difference. Hence, the bubble behaves as an isothermal one. In other words, in this region, α and κ can be treated as unity in this region. Hence, the equation of sound speed in this region is the same as those of Wood (Eq. (4)).

With the increase of the frequency, the relationship among the three typical lengths becomes

$$\sqrt{\frac{\nu}{\omega}} \ll R_0 \ll \sqrt{\frac{D_g}{\omega}}.$$

In this region, the bubble radius is smaller than the thermal diffusion length but is larger than the viscous length. Hence, the effect of viscosity is weakened and there will be relative motion between the bubbles and liquids. At this time, $\alpha = [1 + 2\beta(1-\beta)/\Gamma]$, where Γ is the coefficient of force acting on bubbles.^[17] Generally, the value of Γ is 1. As the bubble radius is still smaller than the thermal diffusion length, κ is still equal to 1 in this region. Finally, the sound speed could be described as^[17]

$$c_m^2 = \frac{[1 + 2\beta(1-\beta)]p_G}{\rho_l \beta(1-\beta)}. \quad (6)$$

In regions with high frequencies, the relationship between the three typical lengths becomes

$$\sqrt{\frac{\nu}{\omega}} \ll \sqrt{\frac{D_g}{\omega}} \ll R_0.$$

In this region, the relative motion will be more and more enforced and the viscous effects will totally disappear. At the same time, the effects of heat transfer between bubbles and liquids will be trivial and the thermodynamics status of the bubble is an adiabatic process. For this case, $\kappa = \gamma$ and $\alpha = [1 + 2\beta(1-\beta)/\Gamma]$. Then the sound speed of the mixture is^[17]

$$c_m^2 = \frac{[1 + 2\beta(1-\beta)]\gamma p_G}{\rho_l \beta(1-\beta)}. \quad (7)$$

In most engineering applications, the void fraction is much smaller than 1. In this situation, ignoring the

terms of order β^2 , Eqs. (6) and (7) can be simplified to^[17]

$$c_m^2 = \frac{(1 + 2\beta)p_G}{\rho_l\beta(1 - \beta)}, \quad (8)$$

$$c_m^2 = \frac{(1 + 2\beta)\gamma p_G}{\rho_l\beta(1 - \beta)}. \quad (9)$$

At low frequencies, the sound speed of wave propagation in the liquids with gas bubbles varies slowly and it can be treated as a constant, which is independent of the frequencies.^[19] For convenience, some simplified formulae (e.g., Refs. [14–16]) have been derived for the calculation of sound speeds at low frequencies in the engineering practice.

Brennen^[14] proposed that the acoustic impedance of the mixed fluid is based on the volume content of each component by its weighted average. For a gas–liquid mixture, with the assumption of ideal gas, the acoustic impedance can be expressed as^[14]

$$\frac{1}{\rho_m c_m^2} = \frac{\beta}{\kappa p_G} + \frac{1 - \beta}{\rho_l c_l^2}. \quad (10)$$

In many applications, $p_G \ll \rho_l c_l^2$ and the second term on the right-hand side of Eq. (10) could be ignored. Hence, Eq. (10) can be simplified to^[14]

$$\frac{1}{c_m^2} = \frac{\beta}{\kappa p_G} [\rho_l(1 - \beta) + \rho_g\beta]. \quad (11)$$

This is the simplified expression of the sound speed in the two-phase bubbly flows. In Eq. (11), if the term $\rho_g\beta$ is ignored, the classic Wood formula (Eq. (4)) will be obtained.

At low frequencies (as $\omega \rightarrow 0$), bubble motion shows the isothermal behavior ($\kappa = 1$). The dispersion relationship is transferred into^[16]

$$\frac{c_l^2}{c_m^2} = 1 + \frac{\beta\rho_l c_l^2}{p_G} \frac{1}{1 - \frac{2\sigma}{3R_0 p_G}}. \quad (12)$$

Disregarding the surface tension in Eq. (12), Eq. (12) becomes^[15]

$$c_m^2 = \frac{c_l^2}{1 + \frac{\beta\rho_l c_l^2}{p_0}}. \quad (13)$$

In Eq. (13), for most cases, $\beta\rho_l c_l^2/p_0 \gg 1$. Hence, the first term in the denominator can be eliminated. Then, Eq. (13) can be simplified to^[16]

$$c_m^2 = \frac{p_0}{\beta\rho_l}. \quad (14)$$

Equations (12)–(14) are only applicable to the case of small void fractions. Furthermore, Eq. (14) could also be obtained from the reduction of Wood's formula (Eq. (4)) by assuming a small void fraction and ignoring surface tension.

In the formulae mentioned above, depending on the frequencies (low, medium and high frequencies), the validity of the formulae could be categorized into three regions in terms of frequency. Now, a quantitative analysis will be performed based on the analysis of the three typical lengths involved. To facilitate analysis, the viscous length (with the order of the $\sqrt{\nu/\omega}$) is expressed by δ_v , and the thermal diffusion length (with the order of the $\sqrt{D_g/\omega}$) is expressed by δ_{th} . The following physical properties are employed: $T_0 = 300$ K, $P_0 = 1.01 \times 10^5$ Pa, $\rho_l = 996.56$ kg/m³, $\sigma = 0.072$ N/m, $\nu = 8.57 \times 10^{-7}$ m²/s, and $D_g = 1.9 \times 10^{-5}$ m²/s.

Figure 1 shows the relationship between three typical characteristic lengths versus the variations of the frequencies with the bubble radius of 100 μ m. As shown in Fig. 1, the line of R_0 and the other two characteristic lengths have two intersections (ϖ_1 and ϖ_2), which are marked using red solid dots in Fig. 1. Here in the case of Fig. 1, $\varpi_1 = 85.7$ rad/s and $\varpi_2 = 1900$ rad/s. Hence, the full parameter zone can be divided into three regions (named as regions 1, 2 and 3) by the above two intersections which correspond to Eqs. (4), (6) and (7), respectively. For the details of each region, researchers are referred to Table 1.

Table 1. Characteristics of three regions defined in this work by comparing three typical lengths.

Frequency	Region	Lengths	Equation
Low frequency	Region 1	$R_0 < \sqrt{\frac{\nu}{\omega}} < \sqrt{\frac{D_g}{\omega}}$	Eq. (4)
Medium frequency	Region 2	$\sqrt{\frac{\nu}{\omega}} < R_0 < \sqrt{\frac{D_g}{\omega}}$	Eq. (6)
High frequency	Region 3	$\sqrt{\frac{\nu}{\omega}} < \sqrt{\frac{D_g}{\omega}} < R_0$	Eq. (7)

The boundary points of the three regions (ϖ_1 and ϖ_2) vary with the bubble radius (referring to Table 2). From Table 2, with the increase of the bubble radius, both ϖ_1 and ϖ_2 decrease remarkably. Furthermore, the values of ϖ_1 and ϖ_2 become much closer with the increase of the bubble radius. Therefore, the aforementioned three regions strongly depend on the bubble radius. For example, when $R_0 = 1$ cm, $\varpi_1 = 0.00857$ rad/s and $\varpi_2 = 0.19$ rad/s. At this time, region 3 occupies the vast majority of the whole parameter zone. Therefore, at nearly all the frequen-

cies, the relative motion between bubbles and liquids is very important and region 3 is the dominant one. For most cases, the classic Wood formula will be no longer valid if $\omega > \varpi_1$.

Based on the formulae given above, the values of sound speed strongly depend on the void fraction. Figure 2 indicates the variations of sound speed versus void fraction in the three regions mentioned above with the bubble radius of 100 μ m. The formulae for the low, medium and high frequencies are Eqs. (4), (6) and (7), also referring to Table 1. As shown in

Fig. 2, the overall trend of the three curves is basically consistent. However, compared with Eqs. (6) and (7), the traditional Wood formula (Eq. (4)) underestimates the sound speed in bubbly liquids in medium and high frequency regions. For example, the minimum sound speeds (at the intermediate void fractions) are 20 m/s, 24 m/s and 29 m/s for the predictions based on Eqs. (4), (6) and (7), respectively. As the cavitation modeling is quite sensitive to the sound speed,^[9,10] the above difference between predictions of three formulae should be emphasized.

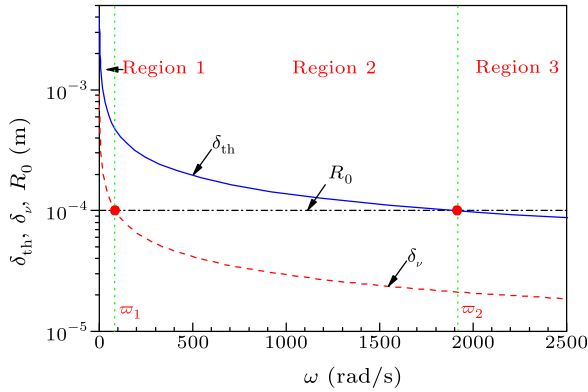


Fig. 1. The variations of bubble radius, the viscous length and the thermal diffusion length versus frequency. In this figure, ϖ_1 and ϖ_2 (referring to red solid dots) are the two intersections between line R_0 and the other two curves of characteristic lengths. Three regions (regions 1, 2 and 3) are defined based on the values of ϖ_1 and ϖ_2 . Here $R_0 = 100 \mu\text{m}$.

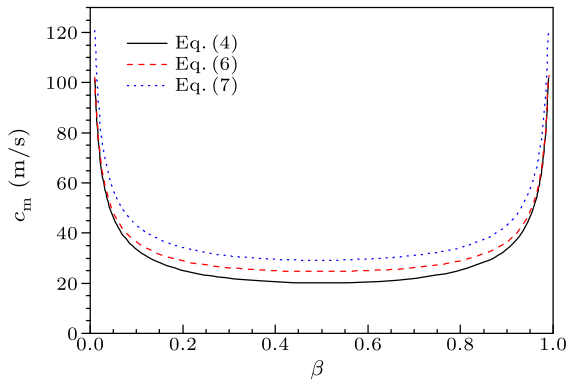


Fig. 2. The variations of the sound speed predicted by Eqs. (4), (6) and (7) versus the void fraction. Here $R_0 = 100 \mu\text{m}$.

Table 2. The variations of boundary points of the three regions (ϖ_1 and ϖ_2) versus the bubble radius.

R_0 (m)	ϖ_1 (rad/s)	ϖ_2 (rad/s)
1.00×10^{-5}	8.57×10^3	1.90×10^5
5.00×10^{-5}	3.43×10^2	7.60×10^3
1.00×10^{-4}	8.57×10^1	1.90×10^3
5.00×10^{-4}	3.43×10^0	7.60×10^1
1.00×10^{-3}	8.57×10^{-1}	1.90×10^1
5.00×10^{-3}	3.43×10^{-2}	7.60×10^{-1}
1.00×10^{-2}	8.57×10^{-3}	1.90×10^{-1}

Now, several simplified formulae of Wood's framework are also compared to show their validities. Figure 3 shows the predictions of sound speed by Wood^[13] (Eq. (4)), Brennen^[14] (Eq. (11)), Ando *et al.*^[15] (Eq. (13)) and Prosperetti^[16] (Eq. (14)) up to void fraction 0.30. Based on Fig. 3, the predictions by Eqs. (4) and (11) are purely identical and the predic-

tions by Eqs. (13) and (14) are purely identical. Here the bubble radius is $100 \mu\text{m}$ and the effects of surface tension could be ignored. With the increase of the void fraction, the errors of predictions by Eqs. (13) and (14) will increase compared with more complete Wood's formula (Eq. (4)). As marked in Fig. 3, for void fraction larger than 0.187, the errors of Eqs. (13) and (14) will exceed 10%.

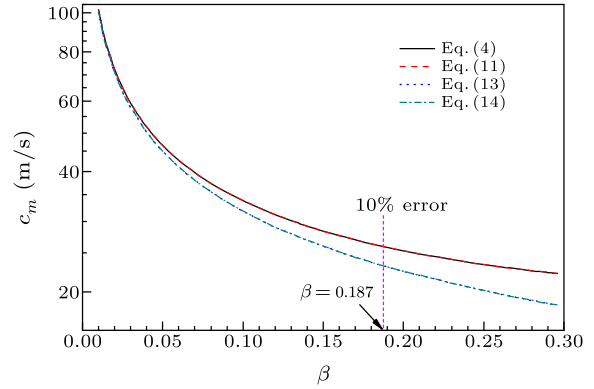


Fig. 3. Comparisons of simplified formulae of sound speed. Formulas are Eqs. (4), (11), (13) and (14). Here $T_0 = 300 \text{ K}$, and $R_0 = 100 \mu\text{m}$.

In conclusion, the importance of the effects of relative motion and thermal diffusion are quantitatively discussed through the analysis of three typical characteristic lengths (δ_v , δ_{th} and R_0). The full parameter zone is divided into three regions with the characteristics of each region given. Our results reveal that the effects of relative motion are of great importance in a wide range of parameters (e.g., large bubble radius or medium/high frequencies). Validities of the formulae are quantitatively defined with the aid of the aforementioned three regions.

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