

# Pinning consensus analysis of multi-agent networks with arbitrary topology\*

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In this paper the pinning consensus of multi-agent networks with arbitrary topology is investigated. Based on the properties of  $M$ -matrix, some criteria of pinning consensus are established for the continuous multi-agent network and the results show that the pinning consensus of the dynamical system depends on the smallest real part of the eigenvalue of the matrix which is composed of the Laplacian matrix of the multi-agent network and the pinning control gains. Meanwhile, the relevant work for the discrete-time system is studied and the corresponding criterion is also obtained. Particularly, the fundamental problem of pinning consensus, that is, what kind of node should be pinned, is investigated and the positive answers to this question are presented. Finally, the correctness of our theoretical findings is demonstrated by some numerical simulated examples.

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## 1. Introduction

In recent years, distributed coordination of multi-agent networks has received increasing attention in many fields including cooperative control of unmanned air vehicles, formation control, flocking, distributed sensor networks, and congestion control in communication networks. One of the critical research problems in distributed coordinated control is how to control all agents of the networks to reach an agreement regarding a certain quantity of interest that depends on the states of all agents.<sup>[1]</sup> This is the so-called consensus problem. Consensus problems have a long history in the field of computer science, particularly in automata theory and distributed computation.<sup>[2]</sup> In the past few years, the consensus problem of multi-agent networks has been extensively investigated as seen in Refs. [1]–[15] and the references therein.

In normal cases, some appropriate controllers are usually required to be designed for regulating a multi-agent network to a consistent trajectory. Of course, it is costly and impractical to control all the nodes in a network. Alternatively, we can adopt the pinning control strategy to drive the network from any initial state to a desired consensus state by applying local control actions to a small fraction of the network nodes. Subsequently, extensive work has been devoted to the study of consensus or synchronization of multi-agent networks and complex networks by using the pinning controller.<sup>[1,16–26]</sup> The pinning consensus problem of multi-agent nonlinear directed networks

was discussed by using the Lie algebra theory.<sup>[1]</sup> In Ref. [16], an optimal pinning control scheme of general complex dynamical networks with linear feedback was investigated. An effective predictive mechanism via pinning control was designed to improve the consensus performance of multi-agent systems.<sup>[17]</sup> In Ref. [18], investigated was the synchronization of directed networks whose coupling matrices were reducible and asymmetrical and obtained was a strong sufficient condition to guarantee the synchronization of the networks. In Ref. [19] it is shown that complex networks with undirected or directed topology with a spanning tree can achieve synchronization by pinning a single controller when the coupling strength is large enough. In Refs. [20] and [21], the pinning consensus of dynamical networks with undirected topology were discussed and in Ref. [22], some low-dimensional pinning criteria for global synchronization of both directed and undirected complex networks were presented. The consensus problem in a multi-agent system with general nonlinear coupling and strongly connected topology was investigated in Ref. [23]. Some mathematical analyses were presented for achieving a lower cost in the synchronization of different star-shaped networks in Ref. [24]. In Ref. [25] studied was the problem of pinning impulsive synchronization for complex networks with directed or undirected but a strongly connected topology. The topologies of the system are also fixed and contain a spanning tree.<sup>[26]</sup>

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However, what we have to point out here is that the coupling configuration matrix is assumed to be symmetric or irreducible in most of the aforementioned literatures, which implies that the topology of the corresponding multi-agent system or complex network is either undirected, strongly connected or owns a directed spanning tree. These conditions are so special that they are not suitable for studying the common problems. Motivated by the related work, and in order to overcome the shortcoming mentioned above, we study the pinning consensus problems of multi-agent networks with arbitrary topology. The main contributions of this paper are presented as follows. Firstly, some criteria of pinning consensus which are suitable for the network with arbitrary topology are established for the continuous multi-agent network based on the properties of  $M$ -matrix and from the results, we know that the smallest real parts of the eigenvalues of the matrix composed of the pinning control gains and the Laplacian matrix of the multi-agent system can measure the pinning consensus of the multi-agent networks. It coincides with one of the conclusion in Ref. [26], but the latter only discussed the complex networks whose topology is directed and owns a spanning tree. Secondly, the research which extends to the discrete-time case is reviewed. Likewise, we also obtain the condition which can guarantee that the discrete multi-agent system with general topology achieves pinning consensus. To the best of our knowledge, there are few research papers dealing with this issue. Finally, we discuss the fundamental problem in pinning control, i.e., what kind of node should be pinned. The novel scheme we propose in this paper is applicable to the multi-agent network with arbitrary topology.

The rest of the paper is organized as follows. In Section 2, some preliminaries and lemmas are given. The main theorems and corollary for pinning consensus of multi-agent networks are addressed in Sections 3. In Section 4 several numerical simulation examples are presented to validate our theoretical results. Conclusions are finally drawn in Section 5.

## 2. Preliminaries

Here, we firstly list some mathematical notations used throughout the paper,  $\mathfrak{R}$  and  $\mathfrak{C}$  denote the sets of real numbers and complex numbers, respectively, and  $\text{Re}(\cdot)$  and  $\text{Im}(\cdot)$  represent the real part and imaginary part of a complex number, respectively,  $I_N$  is an  $N$ -dimensional identity matrix. Denote the transpose of the matrix  $A \in \mathfrak{R}^{n \times n}$  and the vector  $x \in \mathfrak{R}^n$  as  $A^T$  and  $x^T$ , respectively. The inverse of the matrix  $A$  is indicated by  $A^{-1}$ , and  $A_s = (A + A^T)/2$  is its symmetric part. Denote  $A > 0$  ( $A < 0$ ) if  $A$  is positive (negative) definite, and denote its  $i$ -th eigenvalue by  $\lambda_i(A)$ ,  $\otimes$  refers to the Kronecker product.

The information exchange among the nodes in a multi-agent network can be described by an interaction graph. Let

$G = \{V, \varepsilon, A\}$  be a digraph, in which  $V = \{1, \dots, N\}$  is the node set,  $\varepsilon \subseteq V \times V$  is the edge set, and  $A = (a_{ij})_{N \times N}$  is the associated weighted adjacency matrix. An edge of  $G$  is denoted by  $e_{ij} = (v_i, v_j)$ , which means that node  $i$  can receive information from node  $j$ . The entry  $a_{ij} > 0$  if  $e_{ij} \in \varepsilon$ ; otherwise  $a_{ij} = 0$ . In this paper, it is always assumed that  $a_{ii} = 0$  for all  $i \in V$ . The in-degree and out-degree of the node  $i$  are defined as follows:  $\text{deg}_{\text{in}}(i) = \sum_{j=1}^N a_{ji}$ ,  $\text{deg}_{\text{out}}(i) = \sum_{j=1}^N a_{ij}$ , respectively.

The Laplacian matrix  $L = (l_{ij})_{N \times N}$  associated with the adjacency matrix  $A$  is defined as follows:

$$\begin{aligned} l_{ij} &= -a_{ij}, \quad i \neq j; \\ l_{ii} &= \sum_{j=1, j \neq i}^N a_{ij}, \end{aligned} \quad (1)$$

which ensures that  $\sum_{j=1}^N l_{ij} = 0$ . Generally speaking, the Laplacian matrix of a digraph is asymmetric.

Subsequently, some lemmas and definitions are presented from which we need to derive the main results.

**Definition 1**<sup>[27]</sup> A nonsingular matrix  $A = (a_{ij}) \in \mathfrak{R}^{n \times n}$  is called an  $M$ -matrix if  $a_{ij} \leq 0$  whenever  $i \neq j$  and all the elements of  $A^{-1}$  are nonnegative.

**Lemma 1**<sup>[27]</sup> For a nonsingular matrix  $A = (a_{ij}) \in \mathfrak{R}^{n \times n}$  with  $a_{ij} \leq 0$  ( $i \neq j$ ), the following statements are equivalent:

- i)  $A$  is an  $M$ -matrix;
- ii) all eigenvalues of  $A$  have positive real part, i.e.,  $\text{Re}(\lambda_i(A)) > 0$  for all  $i = 1, \dots, n$ ;
- iii) there exists a positive definite diagonal matrix  $\Xi = \text{diag}(\xi_1, \dots, \xi_n) > 0$  such that  $\Xi A + A^T \Xi$  is positive definite, i.e.,  $(\Xi A)_s > 0$ .

**Lemma 2**<sup>[28]</sup> (Gerschgorin Disc Theorem) For matrix  $A = (a_{ij}) \in \mathfrak{R}^{n \times n}$ , all eigenvalues of  $A$  are located in the union of  $n$  discs, i.e.,  $\cup_{i=1}^n \{z \in \mathfrak{C} : |z - a_{ii}| \leq \sum_{j=1, j \neq i}^n |a_{ij}|\}$ ,  $i = 1, \dots, n$ .

**Lemma 3**<sup>[27]</sup> For matrices  $A, B, C, D$  with compatible dimensions, the following equations can be held:

- i)  $(A \otimes B)^T = A^T \otimes B^T$ ;
- ii)  $(A + B) \otimes C = A \otimes C + B \otimes C$ ;
- iii)  $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ .

## 3. Pinning consensus criteria for multi-agent networks

### 3.1. Consensus criteria for pinned continuous multi-agent networks via $M$ -matrix

Consider a linearly coupled multi-agent network composed of  $N$  identical nodes, in which each node is an  $n$ -dimensional dynamical system. The state equations of the system are described by

$$\dot{x}_i(t) = c \sum_{j=1}^N a_{ij} \Gamma [x_j(t) - x_i(t)], \quad i = 1, \dots, N, \quad (2)$$

where  $x_i(t) = [x_{i1}(t), \dots, x_{in}(t)]^T \in \mathfrak{R}^n$  are the state variables of the  $i$ -th node,  $c > 0$  which denotes the coupling strength, the inner coupling matrix  $\Gamma \in \mathfrak{R}^{n \times n}$  and  $\Gamma > 0$ ,  $a_{ij}$  is the  $(i, j)$ -th entry of the adjacency matrix  $A \in \mathfrak{R}^{N \times N}$ .

To achieve the consensus of network (2), some pinning controllers will be added to part of its nodes. Here, to reduce the number of controllers, the pinning strategy is applied to a small fraction  $\delta (0 < \delta < 1)$  of the nodes in network (2). Thus, the pinning controlled network can be described by

$$\dot{x}_i(t) = c \sum_{j=1}^N a_{ij} \Gamma [x_j(t) - x_i(t)] + u_i, i = 1, \dots, N, \quad (3)$$

where

$$u_i = -cd_i \Gamma (x_i(t) - x_0) \in \mathfrak{R}^n, i = 1, \dots, N, \quad (4)$$

are  $n$ -dimensional linear feedback controllers, and the control gains are defined as follows:  $d_i > 0$  if node  $v_i$  is pinned; otherwise,  $d_i = 0$ .  $x_0 \in \mathfrak{R}^n$  denotes the desired equilibrium state. The objective of the pinning consensus is to design controllers for network (3) to achieve  $\lim_{t \rightarrow \infty} |x_i(t) - x_0| = 0, i = 1, \dots, N$ .

**Theorem 1** The pinning controlled multi-agent network (3) with fixed undirected topology can globally achieve consensus if the following holds:

$$-c(L+D) \otimes \Gamma < 0. \quad (5)$$

where  $L$  is the Laplacian matrix associated with the adjacency matrix of the network (3), and  $D = \text{diag}\{d_1, \dots, d_N\}$ ,  $d_i$  is the control gain of node  $i$ ,  $c$  and  $\Gamma$  denote the coupling strength and inner coupling matrix, respectively.

**Proof** Let  $e_i(t) = x_i(t) - x_0$ , from Eqs. (3) and (4), we can have the following error system:

$$\dot{e}_i(t) = c \sum_{j=1}^N a_{ij} \Gamma [e_j(t) - e_i(t)] - cd_i \Gamma e_i(t), \quad i = 1, \dots, N, \quad (6)$$

in view of Eq. (1), we can easily obtain that

$$\dot{e}_i(t) = -c \sum_{j=1}^N l_{ij} \Gamma e_j(t) - cd_i \Gamma e_i(t), \quad i = 1, \dots, N. \quad (7)$$

Let  $e(t) = (e_1^T(t), \dots, e_N^T(t))^T$ , by Lemma 3 and rewrite Eq. (7) in the matrix form as

$$\dot{e}(t) = [-c(L+D) \otimes \Gamma] e(t). \quad (8)$$

By considering the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2} e^T(t) e(t), \quad (9)$$

the derivative of  $V(t)$  along the trajectories of Eq. (6) is as follows:

$$\dot{V}(t) = e^T(t) \dot{e}(t) = e^T(t) [-c(L+D) \otimes \Gamma] e(t), \quad (10)$$

from inequality (5), it is easy to see that  $\dot{V}(t) < 0$ . Therefore, the multi-agent network (3) can globally achieve pinning consensus under the given linear feedback pinning controllers. The proof is completed.

**Remark 1** Condition (5) which is a general criteria to ensure that the pinning consensus coincides with condition (10) in Ref. [20]. But with the increase of the network size, the computation becomes more challenging.

**Corollary 1** Suppose that the inner coupling matrix  $\Gamma \in \mathfrak{R}^{n \times n}$  is positive definite, the controlled multi-agent network (3) with fixed undirected topology can globally achieve pinning consensus if the following condition is satisfied:

$$\min_{1 \leq i \leq N} \text{Re}(\lambda_i(c(L+D))) > 0. \quad (11)$$

Obviously, if the condition (11) holds, as the topology of the system is undirected, then the associated Laplacian matrix  $L$  is symmetrical. So the matrix  $L+D$  is symmetrical too due to  $D$  being a diagonal matrix. Hence, condition (5) can be held, and system (3) can globally reach consensus. The proof is finished.

**Remark 2** Conditions (5) and (11) are very simple, which can provide some guidance in the choice of the control gains. Nevertheless, it is not easy to determine the value of the coupling strength. In the practical application, it is desirable to make the coupling strength as small as possible, in Refs. [20] and [29], the adaptive coupling law was used to achieve this goal. It is an interesting and practical work.

According to Theorem 1, now we will discuss the pinning consensus of multi-agent networks with arbitrary topology.

**Theorem 2** The pinning controlled multi-agent network (3) with arbitrary topology can globally achieve consensus if  $L+D$  is nonsingular and one of the following conditions holds:

$$\text{i) } L+D \text{ is an } M\text{-matrix}; \quad (12)$$

$$\text{ii) } \min_{1 \leq i \leq N} \text{Re}(\lambda_i(L+D)) > 0, \quad (13)$$

where  $L$  is the Laplacian matrix associated with the adjacency matrix of network (3), and  $D = \text{diag}\{d_1, \dots, d_N\}$ ,  $d_i$  is the control gain of node  $i$ .

**Proof** In view of Eqs. (6)–(8), and the proof of Theorem 1, we can construct the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2} e^T(t) \Xi e(t), \quad (14)$$

where  $\Xi \in \mathfrak{R}^{N \times N}$  is a positive definite diagonal matrix. Thus

$$\begin{aligned} \dot{V}(t) &= e^T(t) \Xi \dot{e}(t) = -e^T(t) [\Xi c(L+D)] \otimes \Gamma e(t) \\ &= -ce^T(t) [\Xi(L+D)]_s \otimes \Gamma e(t). \end{aligned} \quad (15)$$

Using Lemma 1, if condition (12) holds, i.e.,  $L+D$  is an  $M$ -matrix, we can know that

$$[\Xi(L+D)]_s > 0. \quad (16)$$

Since  $c$  is a positive constant and  $\Gamma$  is positive definite, and from Eq. (15), we have  $\dot{V}(t) < 0$ . Therefore, error system (8) is globally stable at the origin. Then it follows that pinning-controlled multi-agent networks (3) can asymptotically achieve consensus to the desired equilibrium state  $x_0$ .

Using lemma 1, we can easily know that condition (13) is equivalent to condition (12) if the matrix  $L + D$  is nonsingular. By now the proof is completed.

**Remark 3** Due to the feature of  $M$ -matrix which does not necessarily need to be symmetrical, Theorem 2 exactly establishes the sufficient conditions for the pinning consensus of multi-agent networks with directed or undirected topology. Most of the existing research results are based on the undirected or directed topology which either is strongly connected or owns a directed spanning tree. Evidently, the results of Theorem 2 is more common. Meanwhile, condition (13) is a special case of condition (11), so the smallest real parts of the eigenvalues of the matrix  $L + D$  can measure the pinning consensus of the multi-agent networks. One can see that the reaching of the system pinning consensus need not rely on the coupling strength if condition (12) or (13) holds, which may be very practical.

### 3.2. Consensus criteria for pinned discrete-time multi-agent networks

Although the consensus of discrete multi-agent systems has been extensively studied, in this section we will discuss the consensus problem of discrete multi-agent networks via pinning control.

Suppose that the discrete network consists of  $N$  agents, each of which has the dynamics as follows:

$$x_i(k+1) = x_i(k) + u_i(k), \quad i = 1, \dots, N, \quad (17)$$

where  $x_i \in \mathfrak{R}^n$  and  $u_i \in \mathfrak{R}^n$  denote the state and the control input of  $i$ -th agent, respectively. Doing it in the same way as in the continuous system mentioned above, we add pinning controllers to some of the network nodes. Therefore, the pinning controlled network can be described by

$$\begin{aligned} x_i(k+1) &= x_i(k) + u_i(k), \quad i = 1, \dots, N, \\ u_i(k) &= c \sum_{j=1}^N a_{ij} [x_j(k) - x_i(k)] - cd_i [x_i(k) - x^*], \\ i &= 1, \dots, N, \end{aligned} \quad (18)$$

where  $x^* \in \mathfrak{R}^n$  denotes the desired equilibrium state. The objective of pinning consensus is to design controllers for network (18) to achieve  $\lim_{k \rightarrow \infty} |x_i(k) - x^*| = 0, i = 1, \dots, N$ .

**Theorem 3** The pinned discrete multi-agent network (18) with arbitrary topology can globally achieve consensus if the following conditions hold:

$$1) \quad 0 \leq d_i \leq \frac{2}{c} - 2l_{ii}, \quad i = 1, \dots, N, \quad (19)$$

where  $c$  and  $d_i$  denote the coupling strength and the control gain of node  $i$ , respectively, and  $l_{ii}$  is the diagonal element of the Laplacian matrix associated with the adjacency matrix of network (18).

**Proof** Let  $e_i(k) = x_i(k) - x^*$ , we can easily obtain the following error system:

$$e_i(k+1) = e_i(k) - c \sum_{j=1}^N l_{ij} e_j(k) - cd_i e_i(k), \quad i = 1, \dots, N. \quad (20)$$

Let  $e(k) = (e_1^T(k), \dots, e_N^T(k))^T$ , rewrite Eq. (20) in the matrix form as

$$e(k+1) = [I_N - c(L+D)]e(k). \quad (21)$$

Let  $\lambda_i$  be the  $i$ -th eigenvalue of  $[I_N - c(L+D)]$ . As is well known, system (21) can be asymptotically stable if and only if  $[I_N - c(L+D)]$  has all the eigenvalues with the unit circle, i.e.,  $|\lambda_i| < 1$ .

It follows from Gerschgorin disc theorem that

$$|\lambda_i - [1 - c(d_i + l_{ii})]| \leq c \sum_{j=1, j \neq i}^N |l_{ij}|. \quad (22)$$

In view of Eq. (1), one can obtain

$$|\lambda_i - [1 - c(d_i + l_{ii})]| \leq cl_{ii}. \quad (23)$$

After the calculation of some simple steps, we obtain

$$1 - cd_i - 2cl_{ii} \leq \lambda_i \leq 1 - cd_i. \quad (24)$$

From condition (19), we can easily see that  $|\lambda_i| < 1$  holds. Therefore, system (18) with general topology can globally achieve consensus. That ends the proof.

**Remark 4** It is noted that the condition in Theorem 3 is very mild, which is independent of the network structure and the fraction of the pinned node. From the condition, one can know that the pinning consensus of such a discrete-time dynamical network is completely determined by the coupling strength, and also by the control gains and the weighted adjacency value between agents. In addition, similar to the condition in Theorem 2, inequality (19) is the sufficient conditions for the pinning consensus of discrete multi-agent network with arbitrary topology.

### 3.3. Pinning scheme

In the pinning control of multi-agent networks, how to choose pinned nodes for reaching consensus including what kind of node and how many nodes should be pinned is a fundamental problem. Currently, a lot of noticeable researches focusing on this problem have been done. Such as in Ref. [30] it was indicated that the pinned nodes can be randomly or selectively chosen for undirected networks, and better performance

could be achieved when the most highly connected node are pinned. It is shown in Ref. [31] that the root of the spanning tree should be pinned. In Refs. [22] and [26], it was pointed out that the nodes whose out-degrees are bigger than their in-degrees should be pinned. In Ref. [20], it was found that the nodes with low degrees should be pinned first when the coupling strength is small, which is contrary to the common view that the most highly connected nodes should be pinned first. In Ref. [24], there was an interesting result, showing that a lower cost was achieved by using the control scheme of pinning nodes with a small degree. In Ref. [32], a simply approximate formula for estimating the detailed number of pinning nodes was given. At the same time, presented was an adaptive pinning scheme which can guarantee the synchronization of the network if the information about the exact network degree distribution is known. Alternatively, the random pinning scheme can be used to achieve the pinning synchronization. How to select pinned nodes has been still a challengeable work so far. Subsequently, we will study this problem and give the pinning scheme for the multi-agent network with arbitrary topology. For the sake of discussion, we divide it into the following two cases.

**Case 1** For the continuous networks with connected topology, the node with zero-in-degree should be pinned first. Particularly, when the topology of the system is undirected or strong digraph, we can randomly choose the pinned node. The root of the spanning tree should be pinned first if the continuous network with directed topology owns a spanning tree. Obviously, the latter of the two special pinning strategies coincide with the results in some literatures mentioned above. So now we will focus on explaining the normal case. Supposing that the node  $i$  in the network has no in-degree, based on Theorem 2, the node  $i$  should be pinned which can guarantee that the matrix  $L + D$  is nonsingular. In the following, we will give a counter example to show the details.

For simplicity, suppose that the digraph shown in Fig. 1 is the topology of the multi-agent system (3). We can see that the in-degrees of nodes 3 and 4 are zero, and the Laplacian matrix

$$L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If nodes 3 and 4 are not pinned, we cannot ensure that  $L + D$  is a nonsingular matrix. So the conditions in Theorem 2 will not be necessarily satisfied. Of course, the pinning consensus of the network should not be achieved naturally.

Based on the discussion above, we can know that the randomly pinning control scheme may not ensure the consensus of continuous multi-agent networks.

**Case 2** For the discrete network with connected topology, we can randomly choose the pinning node when condition (19)

is satisfied.

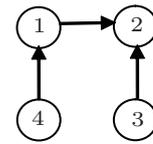


Fig. 1. Directed interaction topology of the network.

#### 4. Simulation examples

In this section, some simulation examples are given to verify the criteria established above.

**Example 1** In this example, we consider the multi-agent network (3) with 20 nodes and the topology as shown in Fig. 2. For simplicity, we assume that each node of the network is a one-dimensional dynamical system, i.e.,  $n = 1$ , and set  $c = 1$  and  $\Gamma = 1$ . Without loss of generality, the initial states of the agents are generated randomly from 1.0 to 15.0, and the value of the desired equilibrium state is set to be 4.0, i.e.,  $x_0 = 4.0$ . That is to say, we want to drive the network from the initial state to the desired consensus state by applying local control actions to a small fraction of the network nodes. In this simulation, we randomly choose some pinned nodes, and the state trajectories of all the agents are shown in Fig. 3. Obviously, the pinning consensus is achieved in all different cases.

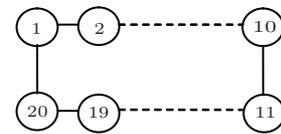


Fig. 2. Undirected interaction topology of the multi-agent network.

**Example 2** In this example, we consider the dynamical network (3) with the directed topology shown in Fig. 4(a). The topology is a digraph and owns a spanning tree. If the values of all the parameters are the same as those in Example 1, we can easily know that the condition in Theorem 2 can be satisfied. The trajectories of all the nodes are shown in Fig. 5. It is not hard to see that the pinning consensus is also reached.

**Example 3** According to Example 2, now we will consider the dynamical network (3) with the directed topology shown in Fig. 4(b). Obviously, node 1 is the zero-in-degree node. In the following research, we conduct a simulation example based on whether the node 1 is pinned. The trajectories of all the nodes are shown in Fig. 6. From the results we can know that the pinning consensus of the system can be achieved when node 1 is chosen to be pinned. Although the consensus of the system can be reached when node 1 is not pinned, the convergence state is not the desired consensus state. So it is further proved that the pinning scheme we have proposed is valid.

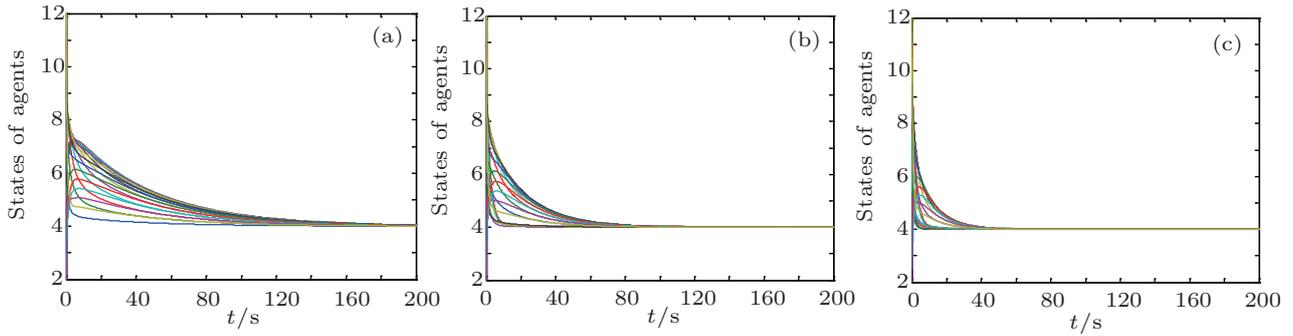


Fig. 3. (color online) Pinning consensus,  $d_i = 3.0$ ,  $x_0 = 4.0$ . (a) node 1 is pinned; (b) nodes 1, 5, 7 are pinned; (c) nodes 1, 5, 7, 9, 11 are pinned.

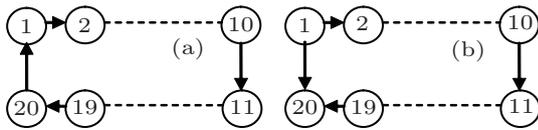


Fig. 4. Directed interaction topology of the multi-agent network.

**Example 4** For convenience, in this example, we also consider the multi-agent network system (18) with the topology shown in Fig. 4(a). For simplicity, we assume that all the corresponding adjacent weights of agents are the same, such as set  $a_{ij} = 0.1$ ,  $i = 1, \dots, 20$ ;  $j = 1, \dots, 20$ , and let the coupling strength  $c = 1$ . The initial states of the agents are generated randomly from 1.0 to 15.0, and the value of the desired equilibrium state is set to be 6.0, that is,  $x^* = 6.0$ . The state trajectories of all the agents are shown in Fig. 7 and we can

find the pinning consensus to be reached.

**Remark** From the results in Examples 1, 2 and 4, we can draw a conclusion that there is a trade-off between the number of pinned nodes and the congestion performance of the multi-agent networks. In normal cases, the more nodes we have chosen, the faster the convergence of the system will be. Normally, we have to consider the cost in practical applications, so usually only an appropriate number of nodes can be selected to be pinned.

**Example 5** On the basis of Example 4, in order to verify the validity of the bound of condition (19), we design the following two cases:

- i)  $a_{12} = 0.1$ ,  $d_1 = 1.8$ ;
- ii)  $a_{12} = 0.2$ ,  $d_1 = 1.8$ .

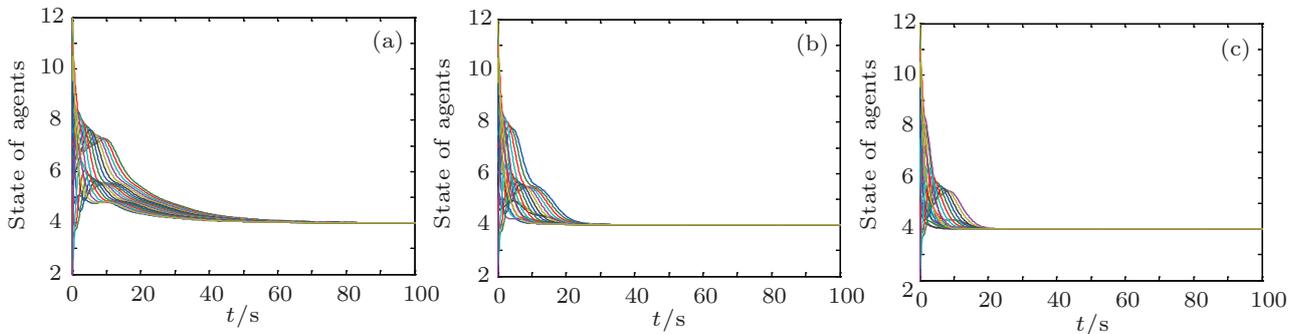


Fig. 5. (color online) Pinning consensus,  $d_i = 3.0$ ,  $x_0 = 4.0$ . (a) Node 1 is pinned; (b) nodes 1, 5, 7 are pinned; (c) nodes 1, 5, 7, 9, 11 are pinned.

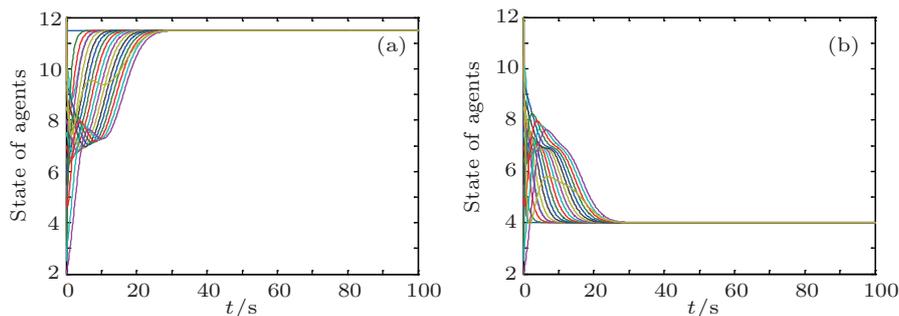
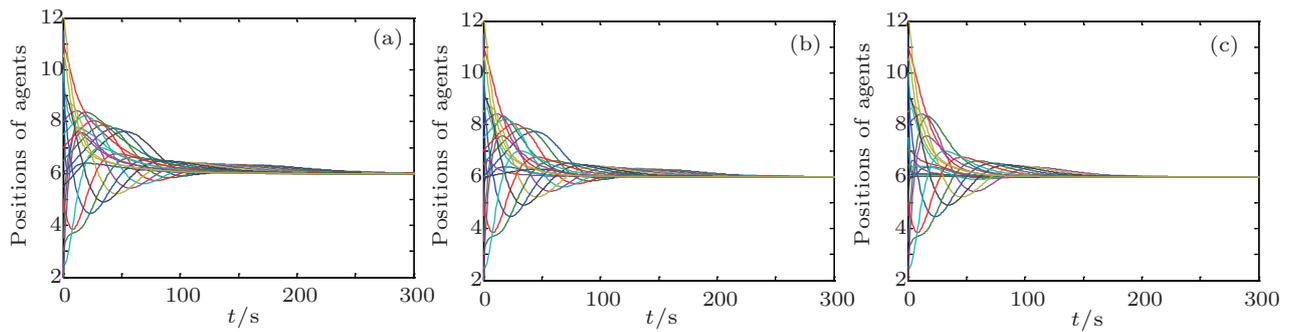
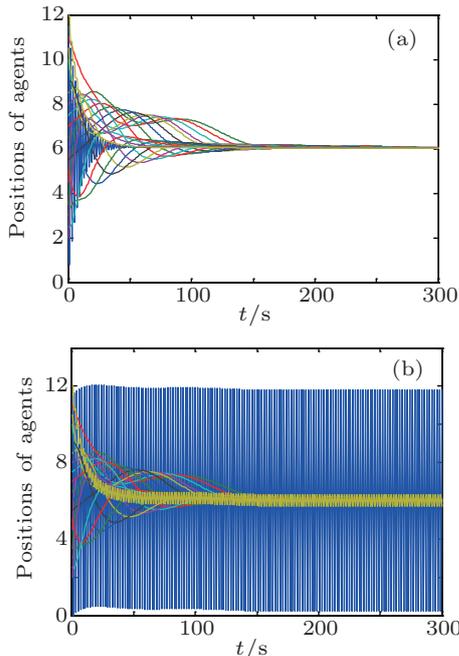


Fig. 6. (color online) Pinning consensus,  $d_i = 3.0$ ,  $x_0 = 4.0$ . (a) Node 1 is not pinned; (b) node 1 is pinned.



**Fig. 7.** (color online) Pinning consensus,  $d_i = 0.5$ ,  $a_{ij} = 0.1$ ,  $x^* = 6.0$ . (a) Node 1 is pinned; (b) nodes 1, 5, and 7 are pinned; (c) nodes 1, 5, 7, 9, and 11 are pinned.



**Fig. 8.** (color online) Pinning consensus when node 1 is pinned,  $x^* = 6.0$  (a)  $a_{ij} = 0.1$ ,  $d_1 = 1.8$ ; (b)  $a_{ij} = 0.2$ ,  $d_1 = 1.8$ .

Evidently, the first case meets condition (19), and the latter one does not. The state trajectories of all the agents are shown in Fig. 8. The experimental results further prove the correctness of our theoretical analysis.

## 5. Conclusion

In this paper, the pinning consensus of multi-agent networks with arbitrary topology is studied. Based on the general criteria for the continuous multi-agent networks with undirected topology, we obtain the criteria for the dynamical system with arbitrary topology by the use of the properties of  $M$ -matrix. It is shown that the pinning consensus of the system depends on the smallest real part of the eigenvalue of the matrix which is the sum of the Laplacian matrix and the diagonal matrix composed of the control gains. Furthermore, we extend the research to the discrete multi-agent networks and obtain the criterion to guarantee that the system achieves pinning consensus. Subsequently, we also discuss the pinning scheme, which is not completely consistent with the results obtained from the aforementioned literature. Finally, some simulations

are provided to verify the correctness of the theoretical analysis. We expect that our findings will be very useful for the current studies of pinning consensus of multi-agent networks. Inspired by the relevant work, such as in Refs. [6] and [13], we will consider the analysis of pinning-controlled networks with switching topologies and nonlinear coupling in the future.

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