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Exact projective solutions of a generalized nonlinear Schrödinger system with variable parameters*

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A direct self-similarity mapping approach is successfully applied to a generalized nonlinear Schrödinger (NLS) system. Based on the known exact solutions of a self-similarity mapping equation, a few types of significant localized excitation with novel properties are obtained by selecting appropriate system parameters. The integrable constraint condition for the generalized NLS system derived naturally here is consistent with the known compatibility condition generated via Painlevé analysis.

Keywords: nonlinear Schrödinger system, self-similarity mapping approach, exact solution, localized excitation

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1. Introduction

It is known that the propagation of electromagnetic waves in nonlinear optical waveguides and ground-state Bose–Einstein condensates (BECs) can be described by a nonlinear Schrödinger (NLS) system[1,2] which is actually one of the fundamental dynamical nonlinear models.[3–5] The generalized NLS system in dimensionless form reads

$$\mathrm{i}\psi_t + f(t)\psi_{xx} + g(t)\psi|\psi|^2 + V(t)x^2\psi = 0, \qquad (1)$$

where $\psi = \psi(x,t)$ is the complex wave envelope in a comoving frame, $f(t)$ is the time-modulated dispersion, $g(t)$ is the nonlinearity, and $V(t)$ is the external harmonic trap potential in BECs. The $V(t)$ is usually absent in nonlinear optical transmission. The subscripts x and t denote the spatial and temporal partial derivative, respectively. These coefficients are often assumed to be real. In the context of many physical fields, the BECs and nonlinear optics provide excellent grounds for exploring nonlinear systems with distributed coefficients. It has been reported that specific dependence of the equation coefficients on time variables can enhance the stability of solutions.[6] Moreover, time-modulated nonlinearity and/or dispersion can facilitate the manipulation of soliton behaviors. These facts have greatly enlarged our knowledge on nonlinear excitations and offered an origin to some important concepts such as nonautonomous solitons[7] and Feschbach resonance, which has been used to control the nonlinearity of matter waves by manipulating the scattering length either in time or space domain, and has led to some proposals of novel nonlinear phenomena. Dispersion management of BECs has also been proposed recently, and many consequent studies followed. In nonlinear optics, both nonlinear management and dispersion management have been used in experiments and theories with temporal or spatial optical solitons, soliton lasers, and ultrafast soliton switches.[7] Furthermore, recent advances of inhomogeneous nonlinear media has generated novel concepts such as the optical similariton.[8] However, the generalized NLS system (1) and/or its similar versions are very difficult to solve because of the presence of the time-dependent dispersion, the nonlinear interaction parameter, and external potential. Up to now, general exact solutions to the generalized NLS system (1) have been rarely found although the knowledge of such exact solutions is very valuable for various purposes. However, some special exact solutions with specific constraint conditions have been obtained, such as the solution obtained by the Lax pair method[7] and that

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obtained by the similarity transformation.[8]

In this paper, we try to offer a general exact projective solution to the generalized NLS system via a mapping approach, which can convert all exact solutions of self-similar well-known models into corresponding solutions of the generalized NLS system. In Section 2, using an extended mapping approach, we will present a quite general exact solution to the generalized NLS system with general constraint conditions. A brief conclusion and discussion will be given in Section 3.

2. An exact projective solution to the generalized NLS system

To obtain exact solutions of a nonlinear partial differential equation (NLPDE), different methods have been applied.[9−11] One of the most powerful methods is the projective approach by establishing and making full advantage of a direct mapping relation between the given NLPDE and another NLPDE or its known solutions sometimes. For example, using the deformation mapping method proposed by Lou and his coworkers,[12] some scholars[13−17] obtained many soliton solutions and periodic solutions of nonlinear models by finding some relations between the exact solutions of the given model and those of the cubic nonlinear Klein–Gordon (NKG) system that has been widely studied previously.[12]

Recently, the usual mapping transformation approach has been extended to find novel localized excitation of a physical model.[18−20] With the help of the idea of mapping transformation and based on the general reduction theory, the mapping algorithm is extended as follows. For a general nonlinear physical system

$$\boldsymbol{P}(\boldsymbol{v}) \equiv \boldsymbol{P}(x_0 = t, x_1, x_2, \ldots, x_n, \boldsymbol{v}, v_{x_i}, v_{x_i x_j}, \ldots), \quad (2)$$

where $\boldsymbol{v} = (v_1, v_2, \ldots, v_q)^{\mathrm{T}}$, $\boldsymbol{P}(\boldsymbol{v}) = (P_1(\boldsymbol{v}), P_2(\boldsymbol{v}), \ldots, P_q(\boldsymbol{v}))^{\mathrm{T}}$, $P_i(\boldsymbol{v})$ are derivatives of polynomials v_i (T indicates the transposition of a matrix). We assume that its solution is in an extended symmetric form

$$\begin{aligned} v_i &= \sum_{j=-N}^{N} \alpha_{ij}(x)\phi^j(\omega(x)), \\ \boldsymbol{x} &\equiv (t, x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, q, \end{aligned} \quad (3)$$

where $\alpha_{ij}(x)$ and $\omega(x)$ are arbitrary functions to be determined. The ϕ is a solution of the Riccati equation: $\phi' = \sigma + \phi^2$, or the generalized Jacobian elliptic equation: $\phi_\xi^2 = \sum_{j=0}^{4} C_j \phi^j$, where σ is a constant, ϕ' is the differentiation of ϕ with respect to ω, and C_j are arbitrary constants. The parameter N is determined by balancing the highest-order nonlinear terms and the highest-order partial terms in the given nonlinear system. Substituting the ansatz (3) together with the related mapping equations into Eq. (2), collecting coefficients of polynomials of ϕ, and then setting each coefficient to be zero, we obtain a set of partial differential equations of $\alpha_{ij}(x)$ and $\omega(x)$. Solving the system of partial differential equations to obtain $\alpha_{ij}(x)$ and $\omega(x)$, substituting the derived results and the solutions of the related mapping equations into Eq. (3), one can obtain many exact solutions to the given nonlinear system.

Motivated by the above ideas, one may assume a self-similar family model as the mapping equation, which has been extensively studied in other literatures.[21,22] For instance, when discussing a nonautonomous Korteweg–de Vries (KdV) system,[23] we may use the classical (1+1)-dimensional KdV equation as the mapping equation since the classical KdV equation has been widely studied. In the following, the (1+1)-dimensional generalized NLS system is selected to illustrate the self-similarity mapping approach (SMA). A general exact solution to the generalized NLS system is derived, which can convert all exact solutions of a self-similar well-known standard NLS equation to the corresponding exact solutions of the generalized NLS system.

The standard NLS equation is regarded as a self-similar mapping equation of the assumed generalized NLS system. Its dimensionless autonomous NLS form is

$$\mathrm{i}\phi_\tau + \alpha\phi_{\varsigma\varsigma} + \beta\phi|\phi|^2 = 0. \quad (4)$$

In addition, Eq.@(4) can be rewritten in a more direct form similar to a mapping equation as

$$\phi_\tau = \mathrm{i}\alpha\phi_{\varsigma\varsigma} + \mathrm{i}\beta\phi|\phi|^2, \quad (5)$$

where α and β are the system parameters, and $\phi = \phi(\varsigma, \tau)$ is a complex wave function of related system dynamics in nonlinear optics. As is known, the nature of the NLS equation has been widely explored via different approaches, and many exact solutions have been reported. For example, when $\alpha = 1/2$ and $\beta = 1$, the NLS equation possesses a canonical bright soliton solution $\phi(\varsigma, \tau) = \mathrm{sech}(\varsigma)\exp(\mathrm{i}\tau/2)$; when $\alpha = -1/2$ and $\beta = 1$, the NLS equation has a fundamental dark soliton solution $\phi(\varsigma, \tau) = \tanh(\varsigma)\exp(\mathrm{i}\tau)$. For more detailed information, one may refer to Ref. [24]. Actually, for the (1+1)-dimensional case, it has been proven that if a physical system can be expressed by a partial differential equation, then one can always find a nonlinear Schrödinger-type equation under some suitable approximations.[25] This is the reason why the (1+1)-dimensional nonlinear Schrödinger system can be successfully used in almost all the branches of physics .

In order to build a direct mapping relation between the generalized NLS system (1) and the standard NLS equation (5), we make an ansatz to the generalized NLS system as follows:

$$\psi(x,t) = \phi(\varsigma(x,t), \tau(t)) A(t) \exp(\mathrm{i}\Omega(x,t)), \tag{6}$$

where $\phi(\varsigma, \tau)$ is an exact complex solution to the standard NLS equation (5), $\varsigma(x,t)$ and $\tau(t)$ are the similarity variables, $\Omega(x,t)$ is the similarity wave phase, and $A(t)$ is a time-dependent function to modulate the wave amplitude to be determined. The reason of factoring out $A(t)$ from $\phi(\varsigma(x,t),\ \tau(t))$ in Eq. (6) is the existence of the coefficient functions of time t in Eq. (1). All the parameters are real differential functions, and should be well chosen to avoid some singularities of the complex wave function $\psi(x,t)$. Substituting Eqs. (5) and (6) into the generalized NLS system (1), vanishing the time-dependent derivative terms, collecting coefficients of polynomials of ϕ and its derivatives, and setting each coefficient as zero, we can obtain the following partial differential equation system:

$$\varsigma_t + 2f\varsigma_x\Omega_x = 0, \tag{7}$$

$$gA^2 - \beta\tau_t = 0, \tag{8}$$

$$f\Omega_x^2 + \Omega_t - Vx^2 = 0, \tag{9}$$

$$f\varsigma_x - \alpha\tau_t = 0, \quad \varsigma_{xx} = 0, \tag{10}$$

$$A_t + fA\Omega_{xx} = 0, \tag{11}$$

After some careful and direct algebraic operations, one can obtain the amplitude, self-similarity variables, and the phase of the complex wave pulse as

$$A(t) = \left(\frac{\beta g}{\alpha f}\right)^{1/2}, \tag{12}$$

$$\varsigma(x,t) = \frac{g}{f}x + C_0\int\frac{g^2}{f}\,\mathrm{d}t + C_1, \tag{13}$$

$$\tau(t) = \frac{1}{\alpha}\int\frac{g^2}{f}\,\mathrm{d}t + C_2, \tag{14}$$

$$\Omega(x,t) = \frac{1}{4f}\left(\frac{f_t}{f} - \frac{g_t}{g}\right)x^2 - \frac{C_0 g}{2f}x - \frac{C_0^2}{4}\int\frac{g^2}{f}\,\mathrm{d}t + C_3, \tag{15}$$

with a constraint condition

$$\frac{g_{tt}}{g} - \frac{f_{tt}}{f} - \frac{2g_t^2}{g^2} + \frac{f_t}{f}\frac{g_t}{g} + \frac{f_t^2}{f^2} + 4fV = 0, \tag{16}$$

where C_0, C_1, C_2, and C_3 are integrable constants.

It is quite interesting to note that the constraint condition (16) is just the completely integrable compatibility condition derived from the Painlevé analysis,[26,27] i.e., a subtle balance condition to keep the generalized NLS system integrable. In view of the management of solitons, Eq. (16) provides an effective way to manipulate soliton dynamics. When any two parameters among $f(t)$, $g(t)$, and $V(t)$ are given, the other one can be adjusted correspondingly according to Eq. (16) to control the coherent dynamics of solitons. Actually, the applications of Eq. (16) have been deeply explored in Ref. [27]. However, the self-similarity mapping transformations (12)–(15) have not been reported. Such transformations are quite systematic in obtaining exact solutions of the generalized NLS system. For the given generalized NLS system or its similar versions, we first check if the coefficients satisfy the constraint condition (16). If it is true, the generalized NLS system can be solved by means of SMA with the standard NLS equation (5). All allowed exact solutions, including canonical solitons, of the standard NLS equation (5) can be converted into the corresponding exact solutions of the generalized NLS system. In this sense, the canonical soliton of the standard NLS equation can be naturally viewed as a seed solution of the corresponding localized solutions of Eq. (1) under the compatibility condition (16).

For example, when $\alpha = 1/2$ and $\beta = 1$, a simple exact solution of the NLS equation (5) reads

$$\phi(\varsigma,\tau) = \phi_0 \mathrm{dn}(k_1\varsigma + k_2\tau + k_3, m) \times \exp[\mathrm{i}(\mu_1\varsigma + \mu_2\tau + \mu_3)], \tag{17}$$

where

$$\phi_0 = k_1, \quad k_2 = -k_1\mu_1, \quad \mu_2 = \frac{(2-m^2)k_1^2 - \mu_1^2}{2}, \tag{18}$$

with k_i and μ_i $(i = 1, 2)$ being arbitrary constants. The $\mathrm{dn}(\cdot, m)$ denotes the dn type elliptic Jacobian function with m being its modulus. This leads to the following family of double periodic wave solutions to the generalized NLS system (1):

$$\psi(x,t) = \phi_0\left(\frac{2g}{f}\right)^{1/2}\mathrm{dn}\left(k_1\frac{g}{f}x + 2k_2\int\frac{g^2}{f}\,\mathrm{d}t + k_3, m\right)\exp[\mathrm{i}\Phi(x,t)], \tag{19}$$

where

$$\Phi(x,t) = \frac{1}{4f}\left(\frac{f_t}{f} - \frac{g_t}{g}\right)x^2 + \mu_1\frac{g}{f}x + 2\mu_2\int\frac{g^2}{f}\,\mathrm{d}t + \mu_3, \tag{20}$$

and the integrable constants C_j ($j = 0,1,2,3$) in Eqs. (12)–(15) are set to be zero for simplicity. In the limit case $m \to 1$, the above double periodic wave solution (19) will reduce to a solitary wave solution under the constraint condition (16) as

$$\psi_{\rm s}(x,t) = \phi_0\left(\frac{2g}{f}\right)^{1/2}\operatorname{sech}\bigg(k_1\frac{g}{f}x + 2k_2\int\frac{g^2}{f}\,{\rm d}t + k_3\bigg)\exp[{\rm i}\varPhi(x,t)]. \quad (21)$$

If $V(t) = 0$, the constraint condition (16) becomes

$$f(t) = g(t)\exp\left[-\gamma\int g(t)\,{\rm d}t\right], \quad (22)$$

where γ is an integrable constant, then the above corresponding exact solutions read

$$\begin{aligned}\psi(x,t) =\ & \sqrt{2}\phi_0 G_\gamma^{1/2}(t){\rm dn}\bigg[k_1G_\gamma(t)x \\ & + 2k_2\int g(t)G_\gamma(t)\,{\rm d}t + k_3, m\bigg] \\ & \times \exp[{\rm i}\varPhi(x,t)], \quad (23)\\ \psi_{\rm s}(x,t) =\ & \sqrt{2}\phi_0 G_\gamma^{1/2}(t){\rm sech}\bigg[k_1G_\gamma(t)x \\ & + 2k_2\int g(t)G_\gamma(t)\,{\rm d}t + k_3\bigg] \\ & \times \exp[{\rm i}\varPhi(x,t)], \quad (24)\end{aligned}$$

where

$$\varPhi(x,t) = -\frac{\gamma}{4}G_\gamma(t)x^2 + \mu_1G_\gamma(t)x + 2\mu_2\int gG_\gamma(t)\,{\rm d}t + \mu_3, \quad (25)$$

with $G_\gamma(t) = \exp\left[\gamma\int g(t)\,{\rm d}t\right]$.

Once the nonlinear parameter $g(t)$ is explicitly given, then all the exact solutions and their related dynamic behaviors of the generalized NLS system can be determined correspondingly. Regarding integrality and readability, some remarks are given below.

Firstly, if $V(t) = 0$ and $f(t) = g(t)$, i.e., $\gamma = 0$ in Eq. (22), the generalized NLS system (1) has canonical soliton solutions regardless of the explicit form of the time-dependent nonlinearity and dispersion. This is because in this case the constraint condition is identified. In this sense, the soliton solution of Eq. (1) is a fundamental canonical soliton. When $f(t) \neq g(t)$, i.e., $\gamma \neq 0$ in Eq. (22), the original balance between nonlinearity and dispersion is destroyed. In this case, the canonical soliton will deform to build a new balance between nonlinearity and dispersion, and the soliton-like solution of Eq. (1) is a deformed canonical soliton or a similariton since the amplitude of the soliton is scaled by the factor $A(t)$. This actually means that if $V(t) = 0$, the exact solitons of the generalized NLS system (1) are canonical or deformed solitons depending on if $f(t)$ is equal to $g(t)$ or not.

Secondly, if $V(t) \neq 0$, the constraint condition (16) indicates that $f(t) \neq g(t)$, which means that the amplitude of the soliton must vary with the factor $A(t)$. This leads to an important phenomenon that there does not exist the canonical and even quasi-canonical matter-wave solitons under the constraint condition (16), which clearly confirms the influence of the dispersion and nonlinear managements on the soliton behaviors.

Finally, it is also interesting to mention that the external trap potential $V(t)$ is absent in the self-similarity mapping transformations (12)–(15). However, the presence of the potential affects the balance between nonlinearity and dispersion due to the constraint condition (16), and builds a deep relation between optical solitons and matter waves.

3. Discussion and conclusion

In summary, the direct self-similarity mapping approach is successfully applied to the generalized nonlinear Schrödinger system. In terms of the known exact solutions of the self-similarity mapping equation, i.e., the standard nonlinear Schröding equation, some significant types of localized excitation with novel properties are revealed correspondingly by selecting appropriate system parameters. The present analysis can be applied to all exact solutions of the generalized nonlinear Schrödinger system. The self-similarity mapping approach provides an effective and systematical way to investigate the nonlinear dynamics of a generalized nonlinear Schrödinger system. As a comparison, it is helpful to refer to some techniques used in previous literatures to find the localized excitation solutions of the generalized NLS equation. The Lax pair analysis is very useful in discussing the integrability conditions. A widely-used approach is the deformation mapping method, which introduces some explicit transformation parameters. These parameters are determined by a set of partial differential equations, which cannot be solved analytically, as emphasized in Ref. [28]. Another similarity transformation reducing the generalized nonlinear Schrödinger equation to a stationary one has also been introduced.[29] By virtue of the Lie point symmetry-group analysis, the generalized nonlinear Schrödinger system or its similar versions can be classified into different classes and each of them can be converted into the corresponding representative equation by using some allowed transformations. As a result, some exact solutions of the representative equation can be transformed into the corresponding solutions of the equations in the same class. However, it was also pointed out in Ref. [30] that in most cases it is still difficult to obtain the exact solutions of these representative equations and the integrability of certain representative equations is not clear. Quite different from the above-mentioned techniques, the present work builds a direct connection between the generalized NLS equation and its standard counterpart, which provides a more systematical way to find exact solutions of the generalized NLS equation. The corresponding transformation formulas are explicit and straightforward. Furthermore, one can naturally derive the integrable constraint condition (16) via the SMA rather than some integrable conditions obtained from Painlevé analysis first. In addition, from the control viewpoint, the self-similarity mapping approach provides an effective way to control the soliton dynamics. To sum up, our present short note is merely an initial work, due to wide potential applications of nonlinearity theory and the requirement to search for more localized excitation approaches. Their applications in reality are worthy of further study.

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References

[1] Strecher K E, Partridge G, Truscott G and Hulet R G 2002 *Phys. Rev. Lett.* **417** 150

[2] Wang Y Y and Zhang J F 2009 *Chin. Phys.* B **18** 1168

[3] Mollenauer L F, Stolen R H and Gordon J P 1980 *Phys. Rev. Lett.* **45** 1095

[4] Zheng C L, Zhang J F, Sheng Z M and Huang W H 2003 *Chin. Phys.* **12** 11

[5] Dai C Q, Wang X G and Zhang J F 2011 *Ann. Phys.* **326** 645

[6] Towers I and Malomed B A 2002 *J. Opt. Soc. Am.* B **19** 537

[7] Serkin V N, Hasegawa A and Belyaeva T L 2007 *Phys. Rev. Lett.* **98** 074102

[8] Ponomarenko S A and Agrawal G P 2006 *Phys. Rev. Lett.* **97** 013901

[9] Wang H and Li B 2011 *Chin. Phys.* B **20** 040203

[10] Xie S Y and Lin J 2010 *Chin. Phys.* B **19** 050201

[11] Wang Y F, Lou S Y and Qian X M 2010 *Chin. Phys.* B **19** 050202

[12] Lou S Y and Ni G J 1989 *J. Math. Phys.* **30** 1614

[13] Zhang Y, Wei W W, Cheng T F and Song Y 2011 *Chin. Phys.* B **20** 110204

[14] Zheng C L and Chen L Q 2004 *Commun. Theor. Phys.* **41** 671

[15] Li H M 2002 *Chin. Phys. Lett.* **19** 745

[16] Fan E G 2003 *J. Phys.* A **36** 7009

[17] Zheng C L, Fang J P and Chen L Q 2005 *Chin. Phys.* **14** 676

[18] Zheng C L and Chen L Q 2008 *Int. J. Mod. Phys.* B **22** 671

[19] Wu H Y, Fei J X and Zheng C L 2010 *Commun. Theor. Phys.* **54** 55

[20] Fei J X and Zheng C L 2011 *Z. Naturforsch.* **66a** 1

[21] Dai C Q, Wang Y Y and Wang X G 2011 *J. Phys.* A **44** 155203

[22] Dai C Q, Chen R P and Zhou G Q 2011 *J. Phys.* B **44** 145401

[23] Tang X Y, Gao Y, Huang F and Lou S Y 2009 *Chin. Phys.* B **18** 4622

[24] Sulem C and Sulem P L 1991 *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse* (New York: Springer-Verlag)

[25] Calogero F, Degasperis A and Xiaoda J 2001 *J. Math. Phys.* **42** 2635

[26] Zhao D, He X G and Luo H G 2009 *Eur. Phys. J.* D **53** 213

[27] Serkin V N, Hasegawa A and Belyaeva T L 2007 *Phys. Rev. Lett.* **98** 074102

[28] Perez-Garcia V M, Torres P J and Harvey V V 2006 *Physica* D **221** 31

[29] Beitia J B, Perez-Garcia V M, Vekslerchik V and Konotop V V 2008 *Phys. Rev. Lett.* **100** 164102

[30] Gagnon L and Winternitz P 1993 *J. Phys.* A **26** 7061