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Mei symmetry and conserved quantities in Kirchhoff thin elastic rod statics*

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We investigate the application of the Mei symmetry analysis in finding conserved quantities for the thin elastic rod statics. By using the Mei symmetry analysis, we have obtained the Jacobi integral and the cyclic integrals for a thin elastic rod with intrinsic twisting for both the cases of circular and non-circular cross sections. Our results can be easily reduced to the results without the intrinsic twisting that have been reported. Through calculation, we find that the Noether symmetry can be more directly and easily used than the Mei symmetry in finding the first integrals for the thin elastic rod. These first integrals will be helpful in the study of exact solutions and stability, as well as the numerical simulation of the elastic rod model for DNA.

Keywords: analytical mechanics, Mei symmetry, conservation laws, elastic rod

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1. Introduction

The Kirchhoff thin elastic rod, as a mechanical model for the DNA supercoil, has been applied to model the configuration and the stability of superhelically constrained DNA.[1–6] On account of the advantage of analytical mechanics methods in research of constrained problems, the elastic rod analytical mechanics frame is constructed[7–10] based on the Kirchhoff dynamic analogy.

Because of the special slender and super-deformation characteristics of the elastic rod model, its equation of motion is strongly nonlinear, which makes its solution difficult to be found. However, the symmetry under the Lie group transformation has its inherent applicability in classifying and reducing differential equations as well as in finding out conservation laws.[11–13] So applying the symmetry to the elastic rod and finding out its conserved quantities via the symmetry analysis will be helpful for its research. Coleman *et al.*[14] introduced the first integrals and the variational principle for the rod dynamics. Maddocks *et al.*[15] gave vector integrals of motion for the rod dynamics and mentioned the corresponding symmetry transformation, but they did not give further discussion. Zhao *et al.*[16] studied the Lie symmetry of a super long elastic rod in the Hamilton form. Jung *et al.*[17] studied a discrete method for special Cosserat elastic rod statics and gave the related Noether theorem. Ding and Fang[18] studied the perturbation of symmetries for super-long elastic slender rods and gave the Noether symmetry, the Lie symmetry, and the Mei symmetry of an elastic slender rod for the circular cross section, but they did not give any more explanation for the application of the symmetry in the elastic rod. In the present paper, we will investigate the application of the Mei symmetry (also called form invariance)[12,13] in finding conserved quantities of Kirchhoff thin elastic rod statics for both circular and non-circular cross sections, and obtain some more general integrals for the elastic rod. Some of the integrals can be obtained more easily by using the symmetry analysis than by other methods. The former first integrals[2] are special cases of our results. The conserved quantities obtained in this paper can be helpful in the numerical simulation for DNA mechanics.

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In this paper, firstly, we introduce the Euler–Lagrange equation of motion for the Kirchhoff elastic rod. Secondly, based on the Kirchhoff dynamic analogy, we give the Mei symmetry, the Mei conserved quantity directly deduced by the Mei symmetry, and the Noether conserved quantity indirectly deduced by the Mei symmetry for the Kirchhoff rod. Thirdly, we study the application of the symmetry analysis in finding the first integrals for the Kirchhoff elastic rod statics.

2. Lagrange equation of thin Kirchhoff elastic rod

Suppose that the elastic rod modeling the DNA is a homogeneous, isotropic, linearly elastic straight rod when unstressed. It is also assumed that no external force or torque is imposed on the rod except at the two ends. We study the cross section of the elastic rod. The motion of the cross section along the arc coordinate describes the configuration of a thin elastic rod. We establish an inertial coordinate system (O–$\xi\eta\zeta$), and the principal axes system (P–xyz) is fixed on the center of the cross section. The configuration of the cross section can be described by position vector $\boldsymbol{r} = \overrightarrow{OP}$, which points from the origin O of the inertial system to the center P of the cross section. The position vector satisfies $\mathrm{d}\boldsymbol{r}/\mathrm{d}s = \boldsymbol{e}_z$, that is the z axis along the tangent direction of the center line of the cross section. Move the origin of (O–$\xi\eta\zeta$) to point P. Let (P–$x_1y_1z_1$) denote the position of (P–$\xi\eta\zeta$) after having the rotation of angle ψ about the ξ axis, and (P–$x_2y_2z_2$) denotes position (P–$x_1y_1z_1$) after the rotation of angle θ about the x_1 axis, and (P–$x_2y_2z_2$) after the rotation of angle φ about the z_2 axis coincides with the principle axis (P–xyz). Here ψ, θ, and φ are the Euler angles of the cross section relative to (P–$\xi\eta\zeta$), which can be seemed as three independent variables to confirm the configuration of the cross section. There exists an integral of principal vector for the elastic rod

$$\boldsymbol{F} = \boldsymbol{F}_0. \tag{1}$$

Take the ζ axis to be parallel with principal vector $\boldsymbol{F}$, the attitude of the elastic rod can be confirmed by the Euler angles. Therefore, the direction cosine α, β, γ of the ζ axis related to the (P–xyz) axis and curvature-twisting vectors ω_i ($i = 1, 2, 3$) can be expressed by using the Euler angles as

$$\alpha = \sin\theta\sin\varphi,\ \beta = \sin\theta\cos\varphi,\ \gamma = \sin\theta, \tag{2}$$

$$\begin{aligned}
\omega_1 &= \frac{\mathrm{d}\psi}{\mathrm{d}s}\sin\theta\sin\varphi + \frac{\mathrm{d}\theta}{\mathrm{d}s}\cos\varphi,\\
\omega_2 &= \frac{\mathrm{d}\psi}{\mathrm{d}s}\sin\theta\sin\varphi - \frac{\mathrm{d}\theta}{\mathrm{d}s}\sin\varphi,\\
\omega_3 &= \frac{\mathrm{d}\psi}{\mathrm{d}s}\cos\theta + \frac{\mathrm{d}\varphi}{\mathrm{d}s}.
\end{aligned} \tag{3}$$

Taking Eqs. (2) and (3) into the Kirchhoff equation of an elastic rod,[1,2] we can obtain the Kirchoff equation in the form of Euler angles.

The goal of classical analytical mechanics is studying the changing laws of the configuration with the independent variable time. For the elastic rod in continuum mechanics, the goal is studying the changing laws of the geometry of the cross section in the arc coordinates. The variational principle of Kirchhoff thin elastic rod statics can be expressed as[2,9]

$$\delta\int_0^L \varGamma + (\boldsymbol{M}_0\cdot\delta\boldsymbol{\phi}_0 + \boldsymbol{M}_L\cdot\delta\boldsymbol{\phi}_L) = 0, \tag{4}$$

where $\delta\boldsymbol{\phi}_0$ and $\delta\boldsymbol{\phi}_L$ denote the virtual angle displacements of the two ends of the elastic rod, respectively. Based on the Kirchhoff dynamic analogy, $\varGamma$ denotes the Lagrangian function of the elastic rod[1]

$$\varGamma = \frac{1}{2}\big[A\omega_1^2 + B\omega_2^2 + C(\omega_3 - \omega_3^0)^2\big] - F\gamma, \tag{5}$$

where A and B are bending rigidities to the x and the y axes, respectively, C is the torsional rigidity to the z axis, F is the internal resultant vector of cross section, and $\boldsymbol{M}_0$ and $\boldsymbol{M}_L$ are the outer moments on the two ends of the elastic rod.

We use the Euler angles, which determine the attitude of cross setion, as the generalized coordinates, $q_1 = \psi$, $q_2 = \theta$, $q_3 = \varphi$. The q'_j denotes the derivative of the generalized coordinate to the arc coordinate and is named the generalized velocity. Taking the generalized coordinates into potential energy density function (5), we can obtain its expression in the generalized coordinate form $\varGamma = \varGamma(s, q_j, q'_j)$, ($j = 1, 2, 3$). We can obtain the Lagrange equation of elastic rod statics from variational principle (4) as

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\partial\varGamma}{\partial q'_j}\Big) - \frac{\partial\varGamma}{\partial q_i} = 0,\ (j = 1, 2, 3). \tag{6}$$

We have used the boundary conditions $(\delta q_i)_{s=0} = 0$ and $(\delta q_i)_{s=L} = 0$. Equation (6) denotes all the motion of the cross section.

3. Mei symmetry and conserved quantities

The Mei symmetry of a mechanical system means that the dynamical function of an equation of motion going through an infinitesimal group transformation still satisfies the original equation of motion.[12] As we know, the Noether symmetry and its conserved quantity have a one-to-one correspondence relation, however, the Mei symmetry does not lead directly to a conserved quantity.[19−27] In the following, we will introduce the Mei symmetry, and the condition under which it can lead to a conserved quantity, and the form of the conserved quantity. Introduce the one-parameter Lie point transformation group in space (s, q_j)

$$\begin{aligned} s^* &= s + \varepsilon \xi_s(s, q_i), \\ q_j^* &= q_j + \varepsilon \eta_j(s, q_i), \quad (i, j = 1, 2, 3), \end{aligned} \tag{7}$$

where ξ_s and η_j are infinitesimal transformation generators, ϵ is an infinitesimal parameter. It has infinitesimal generator

$$X^{(0)} = \xi_s \frac{\partial}{\partial s} + \eta_j \frac{\partial}{\partial q_j}. \tag{8}$$

The first-order prolongation of the infinitesimal generator is

$$X^{(1)} = X^{(0)} + (\eta_j' - \xi_s' q_j') \frac{\partial}{\partial q_j'}. \tag{9}$$

The Lagrange function $\Gamma(s, q_j, q_j')$ becomes $\Gamma(s^*, q_j^*, q_j'^*)$ under infinitesimal transformation (7). Expanding $\Gamma(s^*, q_j^*, q_j'^*)$, we have

$$\begin{aligned} \Gamma(s^*, q_j^*, q_j'^*) &= \Gamma(s, q_j, q_j') + \varepsilon X^{(1)}(\Gamma(s, q_j, q_j')) \\ &\quad + O(\varepsilon^2). \end{aligned} \tag{10}$$

Based on the Kirchhoff dynamic analogy, the variable arc coordinate in the elastic rod statics is the counterpart of variable time in the analytical mechanics. So if the variable arc coordinate takes the role of variable time in results of the Mei symmetry[12,13] in the analytical mechanics, the forms of the results should be the same. Then we can give the definition of the Mei symmetry for the elastic rod statics.

Definition 1 If taking Lagrange function (10) into Eq. (6), the form of Eq. (6) is invariance, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\partial \Gamma^*}{\partial q_j'}\Big) - \frac{\partial \Gamma^*}{\partial q_i} = 0, \ (j = 1, 2, 3), \tag{11}$$

the invariance is called the form invariance, which is also called the Mei symmetry.

From the definition of the Mei symmetry, we can easily obtain the criterion of the Mei symmetry for the elastic rod

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial X^{(1)}(\Gamma)}{\partial q_j'} - \frac{\partial X^{(1)}(\Gamma)}{\partial q_j} = 0. \tag{12}$$

If generators ξ_s and η_j satisfy the following condition:

$$X^{(1)}(\Gamma)\xi_s' + X^{(1)}\big(X^{(1)}(\Gamma)\big) + G_{\mathrm{M}}' = 0, \tag{13}$$

where G_{M} is a gauge function, the Mei symmetry of the system can lead to the Mei conserved quantity

$$\begin{aligned} I_{\mathrm{M}} &= X^{(1)}(\Gamma)\xi_s + \frac{\partial X^{(1)}(\Gamma)}{\partial q_j'}(\eta_j' - q_j'\xi_s') + G_{\mathrm{M}} \\ &= \text{const.} \end{aligned} \tag{14}$$

The Mei symmetry of the elastic rod can also lead to the Noether conserved quantity indirectly. If the Mei symmetry generators ξ_s and η_j satisfy the Noether identity

$$X^{(1)}(\Gamma) + \Gamma\xi_s' + G_{\mathrm{N}}' = 0, \tag{15}$$

where G_{N} is a gauge function, the system has the following Noether conserved quantity:

$$I_{\mathrm{N}} = \Gamma\xi_s + \frac{\partial \Gamma}{\partial q_j'}(\eta_j - q_j'\xi_s) + G_{\mathrm{N}} = \text{const.} \tag{16}$$

4. Application of symmetry in Kirchhoff elastic rod statics

The potential energy density function in the generalized coordinate form for the non-circular cross section is

$$\begin{aligned} \Gamma = \frac{1}{2}\big[& A(q_2'\cos q_3 + q_1'\sin q_2 \sin q_3)^2 \\ & + B(-q_2'\sin q_3 + q_1'\sin q_2\cos q_3)^2 \\ & + C(q_1'\cos q_2 + q_3' - \omega_3^0)^2\big] - F\cos q_2. \end{aligned} \tag{17}$$

Introducing the Hamiltonian with an independent variable in the arc coordinate

$$H = \sum_{i=1}^{3} q_j' \frac{\partial \Gamma}{\partial q_j'} - \Gamma, \tag{18}$$

substituting Eq. (17) into Eq. (18), and omitting the intrinsic twisting ω_3^0, we can obtain the Hamiltonian

$$H = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) + F\gamma. \tag{19}$$

Applying generator vector (9) to Eq. (17), we have

$$
\begin{aligned}
&X^{(1)}(\Gamma)\\
={}&\eta_2\big[A(q_2'\cos q_3+q_1'\sin q_2\sin q_3)q_1'\cos q_2\sin q_3+B(-q_2'\sin q_3+q_1'\sin q_2\cos q_3)q_1'\cos q_2\cos q_3+F\sin q_2\big]\\
&+\eta_3\big[A(q_2'\cos q_3+q_1'\sin q_2\sin q_3)(-q_2'\sin q_3+q_1'\sin q_2\cos q_3)\\
&+B(-q_2'\sin q_3+q_1'\sin q_2\cos q_3)(-q_2'\cos q_3+q_1'\sin q_2\sin q_3)\big]\\
&+(\eta_1'-q_1'\xi_1')\big[A(q_2'\cos q_3+q_1'\sin q_2\sin q_3)\sin q_2\sin q_3+B(-q_2'\sin q_3+q_1'\sin q_2\cos q_3)\sin q_2\cos q_3\\
&+C(q_1'\cos q_2+q_3'-\omega_3^0)\cos q_2\big]+(\eta_2'-q_2'\xi_2')\big[A(q_2'\cos q_3+q_1'\sin q_2\sin q_3)\cos q_3\\
&-B(-q_2'\sin q_3+q_1'\sin q_2\cos q_3)\sin q_3\big]+(\eta_3'-q_3'\xi_3')C(q_1'\cos q_2+q_3'-\omega_3^0).
\end{aligned}\tag{20}
$$

Applying generators

$$X_1^{(1)}=\frac{\partial}{\partial s},\ X_2^{(1)}=\frac{\partial}{\partial q_1},\ X_3^{(1)}=\frac{\partial}{\partial q_3},\tag{21}$$

to Eq. (20), we can straight work out

$$X_1^{(1)}(\Gamma)=X_2^{(1)}(\Gamma)=X_3^{(1)}(\Gamma)=0.\tag{22}$$

Substituting Eq. (22) into the Mei symmetry criterion, we can easily verify that equation (21) is Mei symmetrical. From the Mei structure Eq. (13), we can obtain

$$G_{\mathrm{M}}=0.\tag{23}$$

According to Mei invariant (14), we can only obtain trivial invariant $I_{\mathrm{M}}=0$ from Mei symmetry generator (21).

From the above calculation, we can conclude that the Mei symmetrical generator (21) of the Kirchhoff elastic rod does not correspond to the Mei invariants. But if the generators satisfy the Noether symmetry, we can obtain the Noether invariants indirectly from the Mei symmetry of the Kirchhoff elastic rod. In the following, we will study the Noether conserved quantities indirectly deduced from the Mei symmetry of the Kirchhoff elastic rod.

4.1. Jacobi integrals

Substituting Mei symmetry generator $X_1^{(1)}=\partial/\partial s$ into Noether identity (15), we can work out $G_{\mathrm{N}}=0$. So $X_1^{(1)}=\partial/\partial s$ is Noether symmetrical. We can obtain the Noether conserved quantity for generator $X_1^{(1)}=\partial/\partial s$ as

$$
\begin{aligned}
I_{\mathrm{N}1}={}&-\frac{1}{2}\Big[A(q_2'\cos q_3+q_1'\sin q_2\sin q_3)^2\\
&+B(-q_2'\sin q_3+q_1'\sin q_2\cos q_3)^2\\
&+C(q_1'\cos q_2+q_3'-\omega_3^0)^2+2Cq_1'\cos q_2\omega_3^0\\
&+2Cq_3'\omega_3^0-2C(\omega_3^0)^2\Big]-F\cos q_2.
\end{aligned}\tag{24}
$$

Because $\partial\Gamma/\partial s=0$, so equation (24) can be regarded as the Jacobi integral of equation of motion for the elastic rod. If the intrinsic twisting ω_3^0 is neglected, invariant (24) becomes

$$
\begin{aligned}
I_{N11}={}&-\frac{1}{2}[A(q_2'\cos q_3+q_1'\sin q_2\sin q_3)^2\\
&+B(-q_2'\sin q_3+q_1'\sin q_2\cos q_3)^2\\
&+C(q_1'\cos q_2+q_3')^2]-F\cos q_2\\
={}&-H.
\end{aligned}\tag{25}
$$

Equation (25) can also be written as

$$H=-\Big[\frac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2)+F\gamma\Big],\tag{26}$$

which has the same form ash the result in Ref. [2] but with a negative symbol. Equation (25) is just the negative Hamiltonian of the elastic rod and denotes that the Hamiltonian is invariant along the arc coordinate of the elastic rod. As the work of external forces is zero, equation (25) denotes that the strain potential energy density of the elastic rod is invariant.

For the circular cross section, invariant (24) becomes

$$
\begin{aligned}
I_{N12}={}&-\frac{1}{2}A(q_1'^2\sin q_2^2+q_2'^2)\\
&+\frac{1}{2}C\big[-(q_1'\cos q_2)^2+(q_3'-\omega_3^0)^2\\
&-2q_1'q_3'\cos q_2-2q_3'(q_3'-\omega_3^0)\big]-F\cos q_2.
\end{aligned}\tag{27}
$$

If the intrinsic twisting ω_3^0 is neglected, invariant (27) becomes

$$
\begin{aligned}
I_{N12}'={}&-\frac{1}{2}A(q_1'^2\sin q_2^2+q_2'^2)\\
&+\frac{1}{2}C\big[(q_1'\cos q_2)^2+(q_3')^2+F\cos q_2\big],
\end{aligned}\tag{28}
$$

which is just the negative Hamiltonian of the circular cross section elastic rod, and denotes that the Hamiltonian is invariant along the arc coordinate for the circular cross section elastic rod.

Equations (24) and (25) denote that the difference between the Jacobi integrals of the elastic rod with and without the intrinsic twisting is just a term $C(\omega_3^0)^2$. However, equation (24) denotes that the

Hamiltonian of the elastic rod is no longer invariant along the arc coordinates when the intrinsic twisting is considered. This result is different from the situation that the intrinsic twisting is neglected. Maybe this difference is important for the analyis of the stability of an elastic rod with intrinsic twisting, which will be further studied.

Equation (24) definitely shows that the potential energy density function remains invariant when the sign of the intrinsic twisting is reversed.

Equation (24) can be obtained more easily by using the symmetry analysis than by the other methods. It is a general form of the Jacobi integral for the elastic rod. Equations (27) and (28) have the similar situation.

4.2. Cyclic integrals about ψ

The Mei symmetry infinitesimal generator $X_2^{(1)} = \partial/\partial q_1$ is also Noether symmetrical for $G_{\rm N} = 0$, the corresponding Noether conserved quantity is

$$\begin{aligned} I_{\rm N2} = {} & A(q_2' \cos q_3 + q_1' \sin q_2 \sin q_3) \sin q_2 \sin q_3 \\ & + B(-q_2' \sin q_3 + q_1' \sin q_2 \cos q_3) \sin q_2 \cos q_3 \\ & + C(q_1' \cos q_2 + q_3' - \omega_3^0) \cos q_2, \end{aligned} \tag{29}$$

which can be rewritten in the same form as that in Ref. [2]

$$A\omega_1\alpha + B\omega_2\beta + C(\omega_3 - \omega_3^0)\gamma = M_0. \tag{30}$$

In fact, invariant (29) is the projection of the outer moment on the ζ axis, which denotes that the projection of the principal moment on the ζ axis is invariant along the arc coordinate. Because $\partial\Gamma/\partial q_1 = 0$, so invariant (29) can also be regarded as the cyclic integral to ψ.

For a circular cross section elastic rod, invariant (29) becomes

$$I_{\rm N2}' = A[q_1'^2 \sin q_2^2 + C(q_1' \cos q_2 + q_3') \cos q_2], \tag{31}$$

which can also be regarded as the cyclic integral to ψ.

4.3. Cyclic integrals about φ

For the non-circular cross section, Mei symmetry generator $X_3^{(1)} = \partial/\partial q_3$ is not Noether symmetrical, so it does not correspond to a Noether conserved quantity. But for the circular cross section, $X_3^{(1)} = \partial/\partial q_3$ is Noether symmetrical with gauge function $G_{\rm N} = 0$, so it can deduce the Noether conserved quantity

$$I_{\rm N3} = C(q_1' \cos q_2 + q_3' - \omega_3^0), \tag{32}$$

which is the projection of the principal moment to the cross section in the tangent direction. It also denotes that the torque of the cross section is invariant along the arc coordinates. For $\partial\Gamma/\partial q_3 = 0$, invariant (32) is cyclic integral[2] to φ.

5. Conclusion

The Noether symmetry and the Noether conserved quantity have a one-to-one relation, but the Mei symmetry can only lead to conserved quantities under some conditions.[12]

In the present paper, by using the Mei symmetry and the Noether symmetry analysis methods, we obtain the general form of the Jacobi integral and the cyclic integrals of the Kirchhoff elastic rod for both non-circular and circular cross sections. They can be easily reduced to the results without the intrinsic twisting.[2] Equations (24) and (27) definitely show that the potential energy density function remains invariant for both circular and non-circular cross-section rods when the sign of the intrinsic twisting is reversed.

In the calculation, we find that some first integrals can be obtained easily by using the symmetry analysis methods, which are hard to obtain by using the other methods. The intrinsic curvature and the intrinsic twisting are important for studying the dynamical stability of an elastic rod,[28] so these integrals can be helpful in the numerical simulation of DNA elastic rod mechanical properties and stability.

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