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STATIONARY PROBABILITY DISTRIBUTION FOR COLORED-LOSS-NOISE MODEL OF A SINGLE-MODE DYE LASER*

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We apply the interpolation procedure to calculate the stationary probability distribution of the colored-loss-noise model of a single-mode dye laser operating above the threshold with correlation time τ covering a very wide range. By stochastic Runge-Kutta algorithm, we also carry out numerical simulations of steady-state properties. Comparing the results of the interpolation procedure and the unified colored-noise approximation with simulation results, we find that the agreement between the results of the interpolation procedure and the simulation results is much better than that of the unified colored-noise approximation when correlation time τ covers a range from moderate to large. We conclude that the interpolation procedure really improve the accuracy of predictions for this system.

PACC: 4250; 0540; 4260K

I. INTRODUCTION

The study of dynamical systems perturbed by noise is recurrent in many contexts of physics. In particular, in the context of more realistic models of physical systems, the consideration of noise sources with finite correlation time (i.e., colored noise) has become a subject of current study. One of the typical colored noises is the Ornstein-Uhlenbeck noise. Since a nonlinear system driven by the Ornstein-Uhlenbeck noise is a non-Markovian process, there is no exact Fokker-Planck equation (FPE) for it. Many authors have focused their efforts on obtaining Markovian approximations, trying to capture the essential features of the original non-Markovian problem.[1−6] One particular case is the "unified colored-noise approximation"(UCNA) of Hanggi and collaborators.[3,4] The aim of this approximation can be understood in the following way. The original formulation of the problem is in term of a non-Markovian stochastic differential equation in the relevant variable. However, this problem can be transformed into a Markovian one by extending the number of variables (and equations). The UCNA consists of an adiabatic elimination procedure that allows us to reduce this extended problem to an "effective" Markovian one in the original variable space.

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The ultimate goal of these procedures is to obtain a consistent single variable Fokker-Planck approximation for the probability distribution of the original variable. The UCNA has been justified as a reliable Markovian approximations by means of path integral techniques. The UCNA shows good agreement with the exact numerical results when correlation time $\tau \to 0$ or $\tau \to \infty$. However, it becomes a poor approximation when τ covers a range from moderate to large.

In order to obtain the statistical properties of non-Markovian Processes when τ covers a range from moderate to large, we have focused our attention on extension of the UCNA.[7−9] In 1995, Castro *et al.*[9] presented an extension of the UCNA, through the interpretation of an interpolation procedure (IP) between the white-noise limit and the infinite correlation time limit. Such an interpretation can be more easily seen using a path integral description of the problem. The advantages of this procedure consist in the possibility of devising the interpolating function that best fits a particular set of experimental data, and in this way accurately predicting other relevant functions.

In this paper, we apply the IP to calculate the stationary probability distribution (SPD) for colored-loss-noise model (CLNM) of a single-mode dye laser operating above threshold. Although SPD for CLNM has been obtained by Lin *et al.*[6] using the UCNA, because of being restricted by colored noise, their result just satisfied the small correlation time τ. Here we calculate SPD for CLNM with τ covering a very wide range from $\tau = 0.05$ to 100. By stochastic Runge-Kutta algorithm, we also carry out numerical simulations of SPD. Comparing the results of the IP and the UCNA with simulation results, we find that the agreement between the results of the IP and simulation results is much better than that of the UCNA with τ covering the range from moderate to large.

II. THEORETICAL CALCULATION

The dimensionless equation of motion for the complex field amplitude $E(t)$ for a CLNM of a single-mode dye laser operating above threshold is currently thought to be of the form[2,6]

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left[\frac{A_{\mathrm{g}}}{1+|E|^2} - a_{\mathrm{l}}\right] E + P(t)E \ , \tag{1}$$

where A_{g} and a_{l} are the gain and loss parameters, respectively. The multiplicative noises $P(t) = P_1(t) + \mathrm{i}P_2(t)$ are colored and are assumed to be the Ornstein-Uhlenbeck noises, whose statistical character satisfies the following:

$$< P_i(t) >= 0, \quad < P_i(t)P_j(s) >= \delta_{ij}(D/\tau)\exp(-|t-s|/\tau) \qquad i,j = 1,2 \ , \tag{2}$$

where D denotes the intensity of the noise and τ is the correlation time of the noise.

The change of variables to polar coordinates is according to $E(t) = |E(t)|\exp(\mathrm{i}\varphi)$ and an introduction of the light intensity $I(t) = |E(t)|^2$ is made. The dimensionless equation of intensity $I(t)$ is then obtained from Eq.(1) as follows:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 2\left[\frac{A_{\mathrm{g}}}{1+I} - a_{\mathrm{l}}\right] I + 2IP_1(t) \ , \tag{3}$$

If we transform the variable $I \rightarrow x = \int \mathrm{d}I/2I$, and set

$$f(x) = A_{\mathrm{g}}/(1+I) - a_{\mathrm{l}} \ , \tag{4}$$

Eqs.(3) and (2) are equivalent to a set of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) + P_1(t) \ , \tag{5a}$$

$$\frac{\mathrm{d}P_1}{\mathrm{d}t} = -P_1(t)/\tau + (D^{1/2}/\tau)\xi(t) \ , \tag{5b}$$

$$< \xi(t)\xi(s) > = 2\delta(t-s) \ . \tag{5c}$$

Now we apply the IP to the derivation of the FPE of CLNM. We consider the family of interpolating functions given by[9]

$$\theta[\tau\, f'(x)] = \frac{1}{1+c\tau\, f'(x)} \ . \tag{6}$$

So the corresponding FPE is[9]

$$\begin{aligned}\frac{\partial[P(x,t)]}{\partial t} &= -\frac{\partial}{\partial x}(\{f(x)\theta[\tau\, f'(x)] + D\theta[\tau\, f'(x)]\theta'[\tau\, f'(x)]\}P(x,t)) \\ &\quad + D\frac{\partial^2}{\partial x^2}\{\theta^2[\tau\, f'(x)P(x,t)]\} \\ &= -\frac{\partial}{\partial x}\left(\left\{\frac{f(x)}{1+c\tau\, f'(x)} - \frac{Dc\tau\, f''(x)}{[1+c\tau\, f'(x)]^3}\right\}P(x,t)\right) \\ &\quad + D\frac{\partial^2}{\partial x^2}\left\{\frac{1}{[1+c\tau f'(x)]^2}P(x,t)\right\} \ .\end{aligned} \tag{7}$$

Thus, we obtain for the SPD $P_{\mathrm{st}}(I,t) = P_{\mathrm{st}}(x,t)|\mathrm{d}x/\mathrm{d}I|$ the result

$$\begin{aligned}P_{\mathrm{st}}(I,t) =& NI^{\beta-1}(1+I)^{-\beta_1}[1-4D\beta_1 c\tau I/(1+I)^2] \\ &\times \exp[2D\beta_1^2 c\tau/(1+I)^2 - 4D\beta_1\beta_2 c\tau/(1+I)] \ ,\end{aligned} \tag{8}$$

where $\beta_1 = A_{\mathrm{g}}/(2D)$, $\beta_2 = a_{\mathrm{l}}/(2D)$, $\beta = \beta_1 - \beta_2$, and N is the normalization constant.

When $c = -1$, Eq.(8) reduces to the SPD of the UCNA[6]

$$\begin{aligned}W_{\mathrm{st}}(I,t) =& NI^{\beta-1}(1+I)^{-\beta_1}[1+4D\beta_1\tau I/(1+I)^2] \\ &\times \exp[4D\beta_1\beta_2\tau/(1+I) - 2D\beta_1^2\tau/(1+I)^2] \ .\end{aligned} \tag{9}$$

Ⅲ. DISCUSSION AND CONCLUSION

By using stochastic Runge-Kutta algorithm, we have carried out direct numerical simulations of Eq.(3) with $A_{\mathrm{g}} = 5, a_{\mathrm{l}} = 1$ and $D = 5$ for different values of τ. The circles in Figs.1–5 indicate the result of simulation. When parameters are of the same value, we note that our simulation results by stochastic Runge-Kutta algorithm show exact agreement with numerical simulations of Peacock-Lopez *et al.*, which agrees very well with the experimental results of Zhu, Yu, and Roy for parameter values used in their experiments.[2,10] This indicates that our results of simulation by stochastic Runge-Kutta algorithm are correct.

Now, with the help of Eqs.(8) and (9), we compare the results of the IP and the UCNA with the simulation results. Let us see Figs.1–5. When τ is very small (see Fig.1 τ=0.05), both theoretical results agree very well with the simulations results. If increasing τ(see Figs.2–4 τ=0.5, 5, 50), the results of the UCNA begin to drift off the simulation results. However, at the same time, by regulating parameter c, the results of the IP show better agreement with the simulation results than that of the UCNA. Continually increasing τ to large values (see Fig.5 τ=100), by regulating parameter c, we find remarkable agreement between the results of the IP and the simulation results while the UCNA becomes a poor approximation. From all the above, we conclude that applying the IP to CLNM of a single-mode dye laser can really improve the accuracy of this system for τ covering a range from moderate to large.

In addition, from Fig.1 to Fig.5, we also see that, when increasing τ from small to moderate to large, the extreme point of SPD moves from small intensity I to large intensity I, the peak becomes higher and narrower.

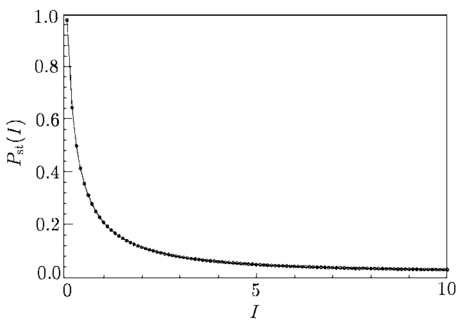

Fig.1. Stationary probability distribution P_{st} shown at $A_g = 5, a_l=1$, and $D = 5$ with $\tau = 0.05, c = -0.95$. We compare the results of the IP (full line) and the UCNA (dashed) with the simulation results (circles).

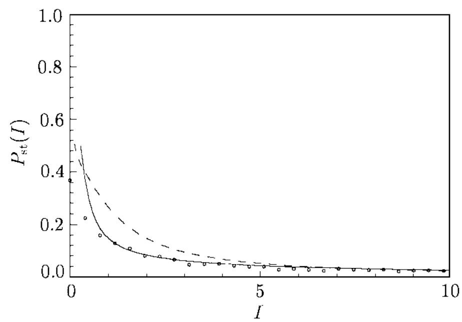

Fig.2. Stationary probability distribution P_{st} shown at $A_g = 5, a_l = 1$, and $D = 5$ with $\tau = 0.5, c = -0.33$. The notations are same as in Fig.1.

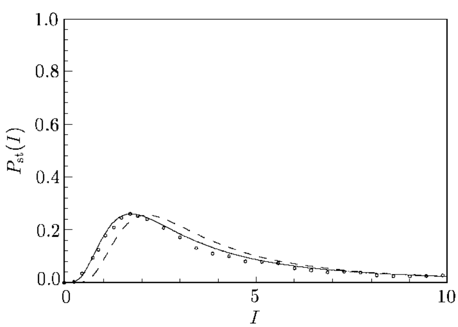

Fig.3. Stationary probability distribution P_{st} shown at $A_g = 5, a_l = 1$, and $D = 5$ with $\tau = 5, c = -0.6$. The notations are same as in Fig.1.

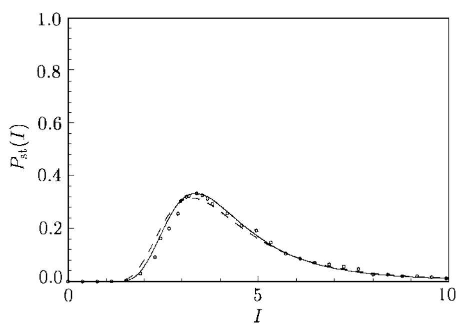

Fig.4. Stationary probability distribution P_{st} shown at $A_g = 5, a_l = 1$, and $D = 5$ with $\tau = 50, c = -1.2$. The notations are same as in Fig.1.

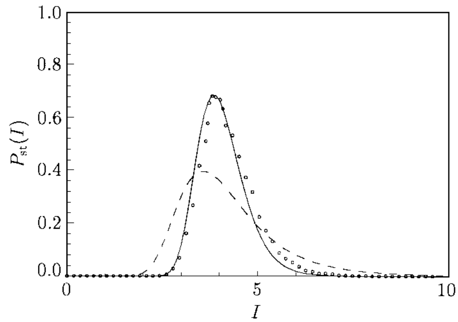

Fig.5. Stationary probability distribution $P_{\rm st}$ shown at $A_{\rm g} = 5, a_{\rm l} = 1$, and $D = 5$ with $\tau = 100, c = -2.5$. The notations are same as in Fig.1.

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